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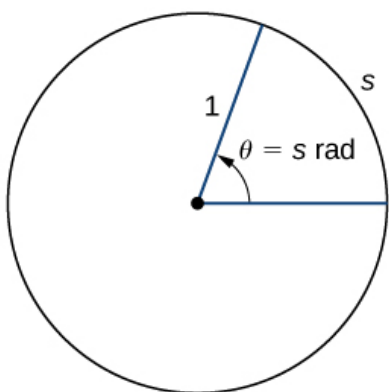
Trigonometric Functions

- Convert angle measures between degrees and radians.
- Recognize the triangular and circular definitions of the basic trigonometric functions.
- Write the basic trigonometric identities.
- Identify the graphs and periods of the trigonometric functions.
- Describe the shift of a sine or cosine graph from the equation of the function.

Trigonometric functions are used to model many phenomena, including sound waves, vibrations of strings, alternating electrical current, and the motion of pendulums. In fact, almost any repetitive, or cyclical, motion can be modeled by some combination of trigonometric functions. In this section, we define the six basic trigonometric functions and look at some of the main identities involving these functions.

Radian Measure

To use trigonometric functions, we first must understand how to measure the angles. Although we can use both radians and degrees, **radians** are a more natural measurement because they are related directly to the unit circle, a circle with radius 1. The radian measure of an angle is defined as follows. Given an angle θ , let s be the length of the corresponding arc on the unit circle ([link](#)). We say the angle corresponding to the arc of length 1 has radian measure 1.



The radian measure of an angle θ is the arc length s of the associated arc on the unit circle.

Since an angle of 360° corresponds to the circumference of a circle, or an arc of length 2π , we conclude that an angle with a degree measure of 360° has a radian measure of 2π . Similarly, we see that 180° is equivalent to π radians. [link](#) shows the relationship between common degree and radian values.

Degrees	Radians	Degrees	Radians
0	0	120	$2\pi/3$
30	$\pi/6$	135	$3\pi/4$
45	$\pi/4$	150	$5\pi/6$
60	$\pi/3$	180	π
90	$\pi/2$		

Common Angles Expressed in Degrees and Radians

Example:

Exercise:

Problem:

Converting between Radians and Degrees

- Express 225° using radians.
- Express $5\pi/3$ rad using degrees.

Solution:

Use the fact that 180° is equivalent to π radians as a conversion factor: $1 = \frac{\pi \text{ rad}}{180^\circ} = \frac{180^\circ}{\pi \text{ rad}}$.

- $225^\circ = 225^\circ \cdot \frac{\pi}{180^\circ} = \frac{5\pi}{4} \text{ rad}$
- $\frac{5\pi}{3} \text{ rad} = \frac{5\pi}{3} \cdot \frac{180^\circ}{\pi} = 300^\circ$

Note:

Exercise:

Problem: Express 210° using radians. Express $11\pi/6$ rad using degrees.

Solution:

$7\pi/6$; 330°

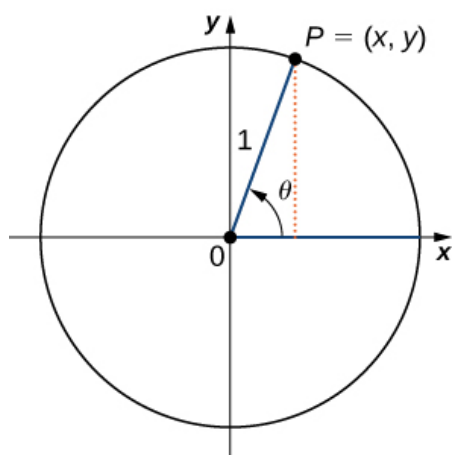
Hint

π radians is equal to 180° .

The Six Basic Trigonometric Functions

Trigonometric functions allow us to use angle measures, in radians or degrees, to find the coordinates of a point on any circle—not only on a unit circle—or to find an angle given a point on a circle. They also define the relationship among the sides and angles of a triangle.

To define the trigonometric functions, first consider the unit circle centered at the origin and a point $P = (x, y)$ on the unit circle. Let θ be an angle with an initial side that lies along the positive x -axis and with a terminal side that is the line segment OP . An angle in this position is said to be in *standard position* ([link](#)). We can then define the values of the six trigonometric functions for θ in terms of the coordinates x and y .



The angle θ is in standard position. The values of the trigonometric functions for θ are defined in terms of the coordinates x and y .

Note:

Definition

Let $P = (x, y)$ be a point on the unit circle centered at the origin O . Let θ be an angle with an initial side along the positive x -axis and a terminal side given by the line segment OP . The **trigonometric functions** are then defined as

Equation:

$$\begin{array}{ll} \sin \theta = y & \csc \theta = \frac{1}{y} \\ \cos \theta = x & \sec \theta = \frac{1}{x} \\ \tan \theta = \frac{y}{x} & \cot \theta = \frac{x}{y} \end{array}$$

If $x = 0$, $\sec \theta$ and $\tan \theta$ are undefined. If $y = 0$, then $\cot \theta$ and $\csc \theta$ are undefined.

We can see that for a point $P = (x, y)$ on a circle of radius r with a corresponding angle θ , the coordinates x and y satisfy

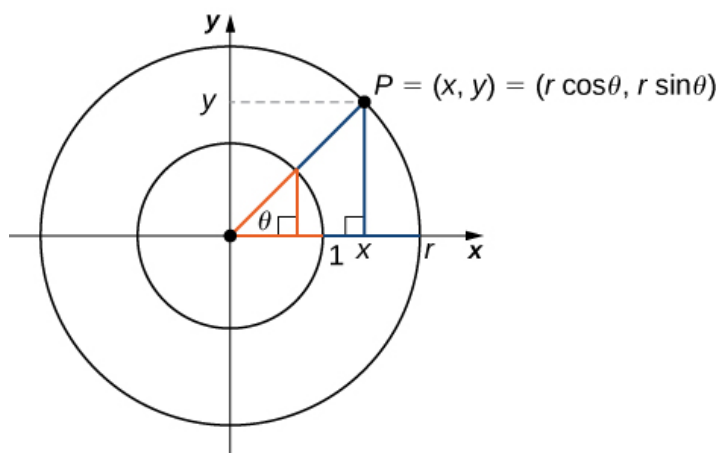
Equation:

$$\cos \theta = \frac{x}{r}$$
$$x = r \cos \theta$$

Equation:

$$\sin \theta = \frac{y}{r}$$
$$y = r \sin \theta.$$

The values of the other trigonometric functions can be expressed in terms of x , y , and r ([link](#)).



For a point $P = (x, y)$ on a circle of radius r , the coordinates x and y satisfy $x = r \cos \theta$ and $y = r \sin \theta$.

[link](#) shows the values of sine and cosine at the major angles in the first quadrant. From this table, we can determine the values of sine and cosine at the corresponding angles in the other quadrants. The values of the other trigonometric functions are calculated easily from the values of $\sin \theta$ and $\cos \theta$.

θ	$\sin \theta$	$\cos \theta$
0	0	1
$\frac{\pi}{6}$	$\frac{1}{2}$	$\frac{\sqrt{3}}{2}$
$\frac{\pi}{4}$	$\frac{\sqrt{2}}{2}$	$\frac{\sqrt{2}}{2}$
$\frac{\pi}{3}$	$\frac{\sqrt{3}}{2}$	$\frac{1}{2}$
$\frac{\pi}{2}$	1	0

Values of $\sin \theta$ and $\cos \theta$ at Major Angles θ in the First Quadrant

Example:

Exercise:

Problem:

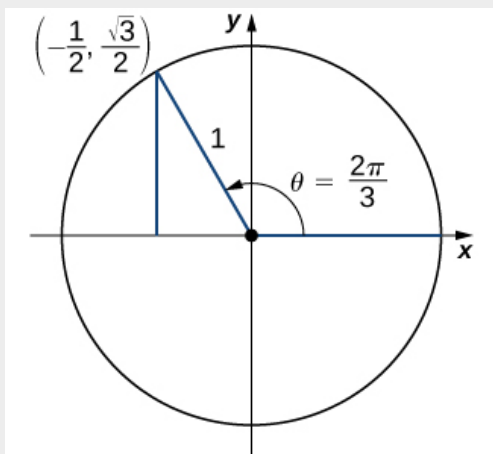
Evaluating Trigonometric Functions

Evaluate each of the following expressions.

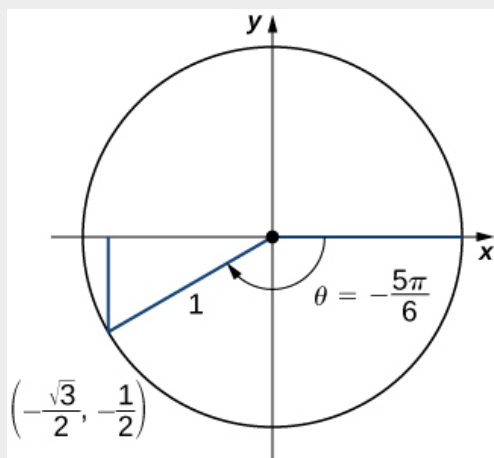
- a. $\sin \left(\frac{2\pi}{3} \right)$
- b. $\cos \left(-\frac{5\pi}{6} \right)$
- c. $\tan \left(\frac{15\pi}{4} \right)$

Solution:

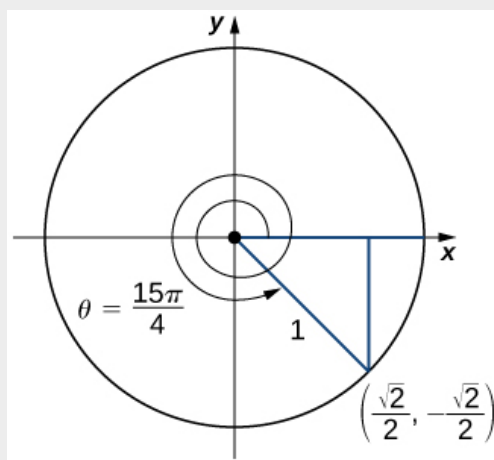
- a. On the unit circle, the angle $\theta = \frac{2\pi}{3}$ corresponds to the point $\left(-\frac{1}{2}, \frac{\sqrt{3}}{2} \right)$. Therefore,
- $$\sin \left(\frac{2\pi}{3} \right) = y = \frac{\sqrt{3}}{2}.$$



- b. An angle $\theta = -\frac{5\pi}{6}$ corresponds to a revolution in the negative direction, as shown. Therefore, $\cos\left(-\frac{5\pi}{6}\right) = x = -\frac{\sqrt{3}}{2}$.



- c. An angle $\theta = \frac{15\pi}{4} = 2\pi + \frac{7\pi}{4}$. Therefore, this angle corresponds to more than one revolution, as shown. Knowing the fact that an angle of $\frac{7\pi}{4}$ corresponds to the point $\left(\frac{\sqrt{2}}{2}, -\frac{\sqrt{2}}{2}\right)$, we can conclude that $\tan\left(\frac{15\pi}{4}\right) = \frac{y}{x} = -1$.



Note:

Exercise:

Problem: Evaluate $\cos(3\pi/4)$ and $\sin(-\pi/6)$.

Solution:

$$\cos(3\pi/4) = -\sqrt{2}/2; \sin(-\pi/6) = -1/2$$

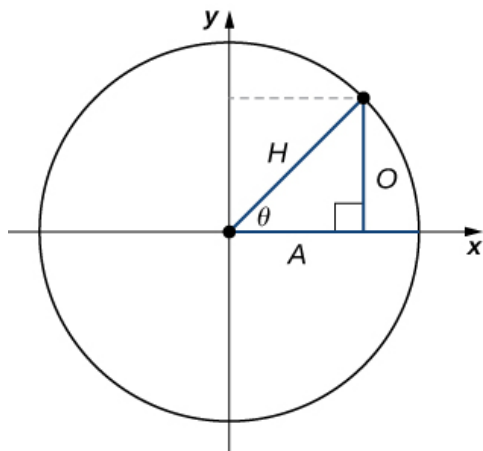
Hint

Look at angles on the unit circle.

As mentioned earlier, the ratios of the side lengths of a right triangle can be expressed in terms of the trigonometric functions evaluated at either of the acute angles of the triangle. Let θ be one of the acute angles. Let A be the length of the adjacent leg, O be the length of the opposite leg, and H be the length of the hypotenuse. By inscribing the triangle into a circle of radius H , as shown in [\[link\]](#), we see that A , H , and O satisfy the following relationships with θ :

Equation:

$$\begin{aligned}\sin \theta &= \frac{O}{H} & \csc \theta &= \frac{H}{O} \\ \cos \theta &= \frac{A}{H} & \sec \theta &= \frac{H}{A} \\ \tan \theta &= \frac{O}{A} & \cot \theta &= \frac{A}{O}\end{aligned}$$



By inscribing a right triangle in a circle, we can express the ratios of the side lengths in terms of the trigonometric functions evaluated at θ .

Example:

Exercise:

Problem:

Constructing a Wooden Ramp

A wooden ramp is to be built with one end on the ground and the other end at the top of a short staircase. If the top of the staircase is 4 ft from the ground and the angle between the ground and the ramp is to be 10° , how long does the ramp need to be?

Solution:

Let x denote the length of the ramp. In the following image, we see that x needs to satisfy the equation $\sin(10^\circ) = 4/x$. Solving this equation for x , we see that $x = 4/\sin(10^\circ) \approx 23.035$ ft.



Note:

Exercise:

Problem:

A house painter wants to lean a 20-ft ladder against a house. If the angle between the base of the ladder and the ground is to be 60° , how far from the house should she place the base of the ladder?

Solution:

10 ft

Hint

Draw a right triangle with hypotenuse 20.

Trigonometric Identities

A **trigonometric identity** is an equation involving trigonometric functions that is true for all angles θ for which the functions are defined. We can use the identities to help us solve or simplify equations. The main trigonometric identities are listed next.

Note:

Rule: Trigonometric Identities

Reciprocal identities

Equation:

$$\begin{aligned}\tan \theta &= \frac{\sin \theta}{\cos \theta} & \cot \theta &= \frac{\cos \theta}{\sin \theta} \\ \csc \theta &= \frac{1}{\sin \theta} & \sec \theta &= \frac{1}{\cos \theta}\end{aligned}$$

Pythagorean identities

Equation:

$$\sin^2 \theta + \cos^2 \theta = 1 \quad 1 + \tan^2 \theta = \sec^2 \theta \quad 1 + \cot^2 \theta = \csc^2 \theta$$

Addition and subtraction formulas

Equation:

$$\sin (\alpha \pm \beta) = \sin \alpha \cos \beta \pm \cos \alpha \sin \beta$$

Equation:

$$\cos (\alpha \pm \beta) = \cos \alpha \cos \beta \mp \sin \alpha \sin \beta$$

Double-angle formulas

Equation:

$$\sin (2\theta) = 2 \sin \theta \cos \theta$$

Equation:

$$\cos (2\theta) = 2 \cos^2 \theta - 1 = 1 - 2 \sin^2 \theta = \cos^2 \theta - \sin^2 \theta$$

Example:

Exercise:

Problem:

Solving Trigonometric Equations

For each of the following equations, use a trigonometric identity to find all solutions.

a. $1 + \cos (2\theta) = \cos \theta$

b. $\sin (2\theta) = \tan \theta$

Solution:

a. Using the double-angle formula for $\cos(2\theta)$, we see that θ is a solution of

Equation:

$$1 + \cos(2\theta) = \cos \theta$$

if and only if

Equation:

$$1 + 2\cos^2\theta - 1 = \cos\theta,$$

which is true if and only if

Equation:

$$2\cos^2\theta - \cos\theta = 0.$$

To solve this equation, it is important to note that we need to factor the left-hand side and not divide both sides of the equation by $\cos\theta$. The problem with dividing by $\cos\theta$ is that it is possible that $\cos\theta$ is zero. In fact, if we did divide both sides of the equation by $\cos\theta$, we would miss some of the solutions of the original equation. Factoring the left-hand side of the equation, we see that θ is a solution of this equation if and only if

Equation:

$$\cos\theta(2\cos\theta - 1) = 0.$$

Since $\cos\theta = 0$ when

Equation:

$$\theta = \frac{\pi}{2}, \frac{\pi}{2} \pm \pi, \frac{\pi}{2} \pm 2\pi, \dots,$$

and $\cos\theta = 1/2$ when

Equation:

$$\theta = \frac{\pi}{3}, \frac{\pi}{3} \pm 2\pi, \dots \text{ or } \theta = -\frac{\pi}{3}, -\frac{\pi}{3} \pm 2\pi, \dots,$$

we conclude that the set of solutions to this equation is

Equation:

$$\theta = \frac{\pi}{2} + n\pi, \theta = \frac{\pi}{3} + 2n\pi, \text{ and } \theta = -\frac{\pi}{3} + 2n\pi, n = 0, \pm 1, \pm 2, \dots$$

- b. Using the double-angle formula for $\sin(2\theta)$ and the reciprocal identity for $\tan(\theta)$, the equation can be written as

Equation:

$$2\sin\theta\cos\theta = \frac{\sin\theta}{\cos\theta}.$$

To solve this equation, we multiply both sides by $\cos\theta$ to eliminate the denominator, and say that if θ satisfies this equation, then θ satisfies the equation

Equation:

$$2\sin\theta\cos^2\theta - \sin\theta = 0.$$

However, we need to be a little careful here. Even if θ satisfies this new equation, it may not satisfy the original equation because, to satisfy the original equation, we would need to be able to divide both sides of the equation by $\cos\theta$. However, if $\cos\theta = 0$, we cannot divide both sides of the equation by $\cos\theta$. Therefore, it is possible that we may arrive at extraneous solutions. So, at the end, it is important to check for extraneous solutions.

Returning to the equation, it is important that we factor $\sin\theta$ out of both terms on the left-hand side instead of dividing both sides of the equation by $\sin\theta$. Factoring the left-hand side of the equation, we can rewrite this equation as

Equation:

$$\sin\theta(2\cos^2\theta - 1) = 0.$$

Therefore, the solutions are given by the angles θ such that $\sin\theta = 0$ or $\cos^2\theta = 1/2$. The solutions of the first equation are $\theta = 0, \pm\pi, \pm2\pi, \dots$. The solutions of the second equation are $\theta = \pi/4, (\pi/4) \pm (\pi/2), (\pi/4) \pm \pi, \dots$. After checking for extraneous solutions, the set of solutions to the equation is

Equation:

$$\theta = n\pi \quad \text{and} \quad \theta = \frac{\pi}{4} + \frac{n\pi}{2}, n = 0, \pm 1, \pm 2, \dots$$

Note:

Exercise:

Problem: Find all solutions to the equation $\cos(2\theta) = \sin\theta$.

Solution:

$$\theta = \frac{3\pi}{2} + 2n\pi, \frac{\pi}{6} + 2n\pi, \frac{5\pi}{6} + 2n\pi \text{ for } n = 0, \pm 1, \pm 2, \dots$$

Hint

Use the double-angle formula for cosine.

Example:

Exercise:

Problem:

Proving a Trigonometric Identity

Prove the trigonometric identity $1 + \tan^2\theta = \sec^2\theta$.

Solution:

We start with the identity

Equation:

$$\sin^2\theta + \cos^2\theta = 1.$$

Dividing both sides of this equation by $\cos^2\theta$, we obtain

Equation:

$$\frac{\sin^2\theta}{\cos^2\theta} + 1 = \frac{1}{\cos^2\theta}.$$

Since $\sin\theta/\cos\theta = \tan\theta$ and $1/\cos\theta = \sec\theta$, we conclude that

Equation:

$$\tan^2\theta + 1 = \sec^2\theta.$$

Note:**Exercise:**

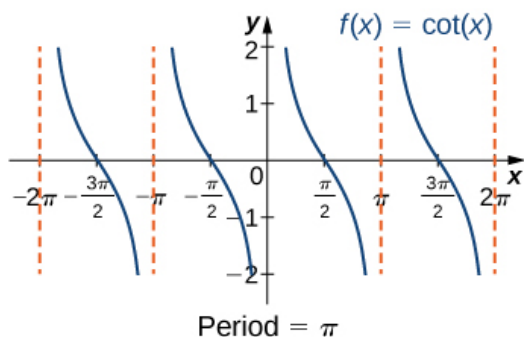
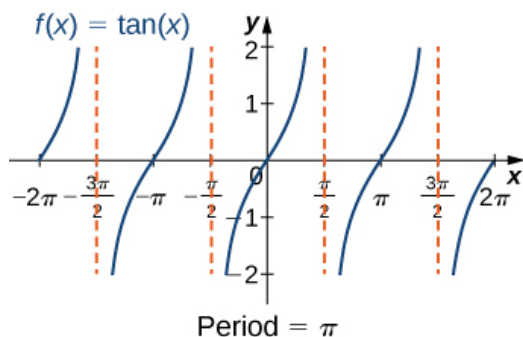
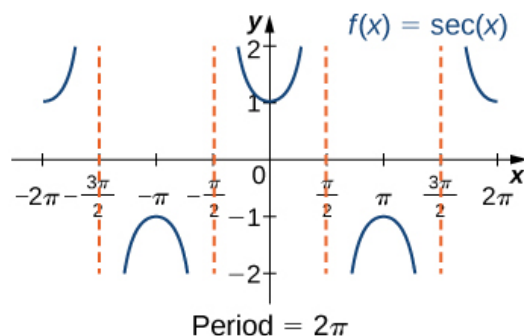
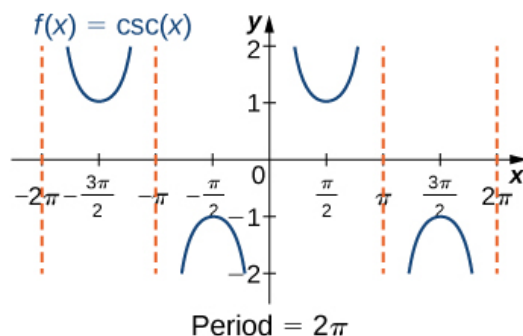
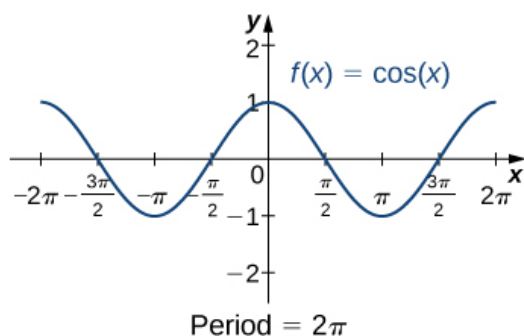
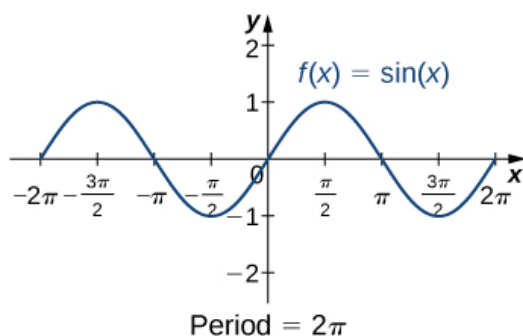
Problem: Prove the trigonometric identity $1 + \cot^2\theta = \csc^2\theta$.

Hint

Divide both sides of the identity $\sin^2\theta + \cos^2\theta = 1$ by $\sin^2\theta$.

Graphs and Periods of the Trigonometric Functions

We have seen that as we travel around the unit circle, the values of the trigonometric functions repeat. We can see this pattern in the graphs of the functions. Let $P = (x, y)$ be a point on the unit circle and let θ be the corresponding angle. Since the angle θ and $\theta + 2\pi$ correspond to the same point P , the values of the trigonometric functions at θ and at $\theta + 2\pi$ are the same. Consequently, the trigonometric functions are **periodic functions**. The period of a function f is defined to be the smallest positive value p such that $f(x + p) = f(x)$ for all values x in the domain of f . The sine, cosine, secant, and cosecant functions have a period of 2π . Since the tangent and cotangent functions repeat on an interval of length π , their period is π ([link](#)).



The six trigonometric functions are periodic.

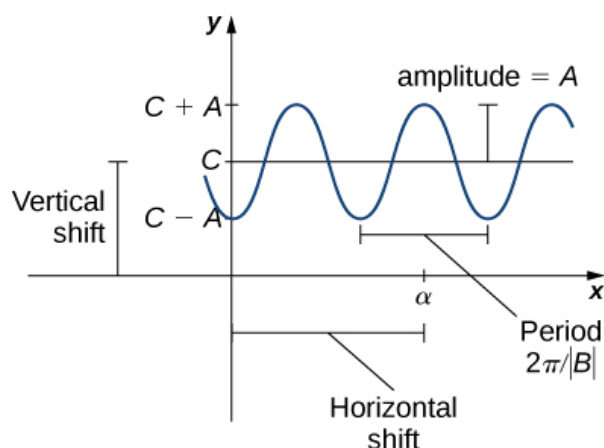
Just as with algebraic functions, we can apply transformations to trigonometric functions. In particular, consider the following function:

Equation:

$$f(x) = A \sin(B(x - \alpha)) + C.$$

In [\[link\]](#), the constant α causes a horizontal or phase shift. The factor B changes the period. This transformed sine function will have a period $2\pi/|B|$. The factor A results in a vertical stretch by a factor of $|A|$. We say $|A|$ is the “amplitude of f .” The constant C causes a vertical shift.

$$f(x) = A\sin(B(x - \alpha)) + C$$



A graph of a general sine function.

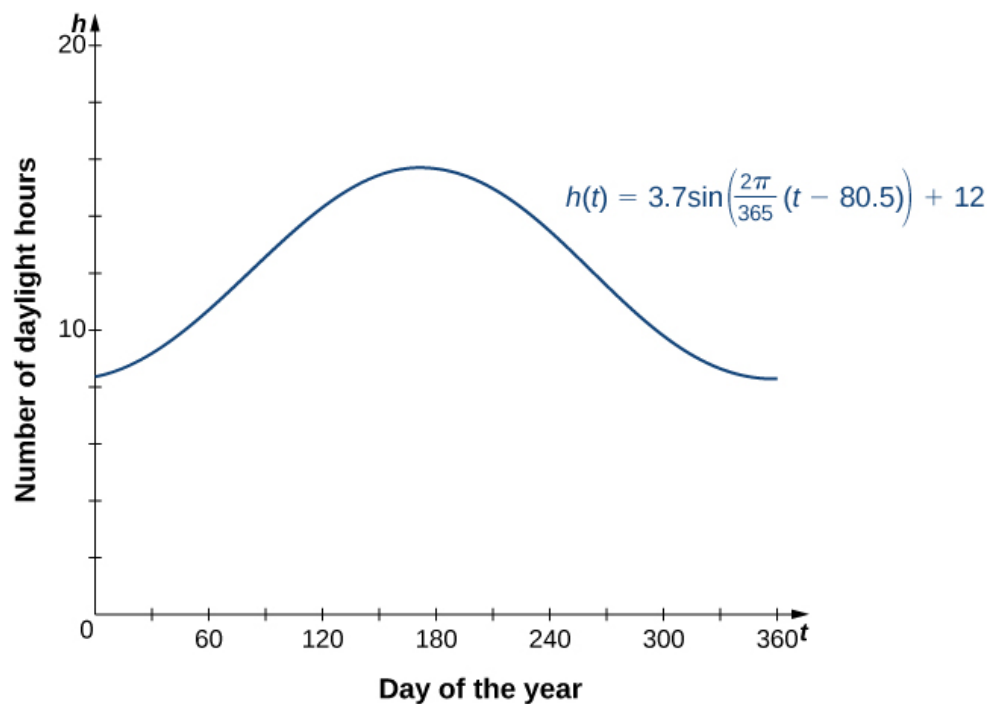
Notice in [\[link\]](#) that the graph of $y = \cos x$ is the graph of $y = \sin x$ shifted to the left $\pi/2$ units. Therefore, we can write $\cos x = \sin(x + \pi/2)$. Similarly, we can view the graph of $y = \sin x$ as the graph of $y = \cos x$ shifted right $\pi/2$ units, and state that $\sin x = \cos(x - \pi/2)$.

A shifted sine curve arises naturally when graphing the number of hours of daylight in a given location as a function of the day of the year. For example, suppose a city reports that June 21 is the longest day of the year with 15.7 hours and December 21 is the shortest day of the year with 8.3 hours. It can be shown that the function

Equation:

$$h(t) = 3.7 \sin\left(\frac{2\pi}{365}(t - 80.5)\right) + 12$$

is a model for the number of hours of daylight h as a function of day of the year t ([\[link\]](#)).



The hours of daylight as a function of day of the year can be modeled by a shifted sine curve.

Example:

Exercise:

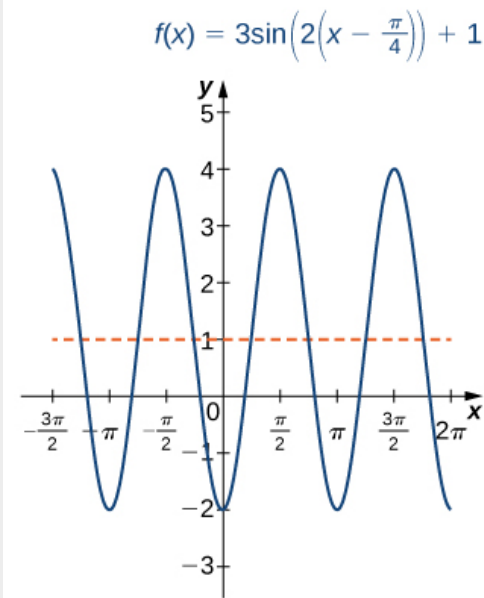
Problem:

Sketching the Graph of a Transformed Sine Curve

Sketch a graph of $f(x) = 3 \sin\left(2\left(x - \frac{\pi}{4}\right)\right) + 1$.

Solution:

This graph is a phase shift of $y = \sin(x)$ to the right by $\pi/4$ units, followed by a horizontal compression by a factor of 2, a vertical stretch by a factor of 3, and then a vertical shift by 1 unit. The period of f is π .



Note:

Exercise:

Problem:

Describe the relationship between the graph of $f(x) = 3\sin(4x) - 5$ and the graph of $y = \sin(x)$.

Solution:

To graph $f(x) = 3\sin(4x) - 5$, the graph of $y = \sin(x)$ needs to be compressed horizontally by a factor of 4, then stretched vertically by a factor of 3, then shifted down 5 units. The function f will have a period of $\pi/2$ and an amplitude of 3.

Hint

The graph of f can be sketched using the graph of $y = \sin(x)$ and a sequence of three transformations.

Key Concepts

- Radian measure is defined such that the angle associated with the arc of length 1 on the unit circle has radian measure 1. An angle with a degree measure of 180° has a radian measure of π rad.
- For acute angles θ , the values of the trigonometric functions are defined as ratios of two sides of a right triangle in which one of the acute angles is θ .

- For a general angle θ , let (x, y) be a point on a circle of radius r corresponding to this angle θ . The trigonometric functions can be written as ratios involving x , y , and r .
- The trigonometric functions are periodic. The sine, cosine, secant, and cosecant functions have period 2π . The tangent and cotangent functions have period π .

Key Equations

- **Generalized sine function**

$$f(x) = A \sin(B(x - \alpha)) + C$$

For the following exercises, convert each angle in degrees to radians. Write the answer as a multiple of π .

Exercise:

Problem: 240°

Solution:

$$\frac{4\pi}{3} \text{ rad}$$

Exercise:

Problem: 15°

Exercise:

Problem: -60°

Solution:

$$\frac{-\pi}{3}$$

Exercise:

Problem: -225°

Exercise:

Problem: 330°

Solution:

$$\frac{11\pi}{6} \text{ rad}$$

For the following exercises, convert each angle in radians to degrees.

Exercise:

Problem: $\frac{\pi}{2} \text{ rad}$

Exercise:

Problem: $\frac{7\pi}{6}$ rad

Solution:

210°

Exercise:

Problem: $\frac{11\pi}{2}$ rad

Exercise:

Problem: -3π rad

Solution:

-540°

Exercise:

Problem: $\frac{5\pi}{12}$ rad

Evaluate the following functional values.

Exercise:

Problem: $\cos\left(\frac{4\pi}{3}\right)$

Solution:

-0.5

Exercise:

Problem: $\tan\left(\frac{19\pi}{4}\right)$

Exercise:

Problem: $\sin\left(-\frac{3\pi}{4}\right)$

Solution:

$-\frac{\sqrt{2}}{2}$

Exercise:

Problem: $\sec\left(\frac{\pi}{6}\right)$

Exercise:

Problem: $\sin\left(\frac{\pi}{12}\right)$

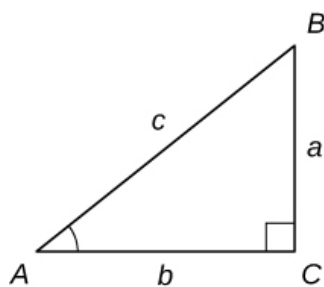
Solution:

$$\frac{\sqrt{3}-1}{2\sqrt{2}}$$

Exercise:

Problem: $\cos\left(\frac{5\pi}{12}\right)$

For the following exercises, consider triangle ABC , a right triangle with a right angle at C . a. Find the missing side of the triangle. b. Find the six trigonometric function values for the angle at A . Where necessary, round to one decimal place.

**Exercise:**

Problem: $a = 4, c = 7$

Solution:

$$\text{a. } b = 5.7 \quad \text{b. } \sin A = \frac{4}{7}, \cos A = \frac{5.7}{7}, \tan A = \frac{4}{5.7}, \csc A = \frac{7}{4}, \sec A = \frac{7}{5.7}, \cot A = \frac{5.7}{4}$$

Exercise:

Problem: $a = 21, c = 29$

Exercise:

Problem: $a = 85.3, b = 125.5$

Solution:

$$\text{a. } c = 151.7 \quad \text{b. } \sin A = 0.5623, \cos A = 0.8273, \tan A = 0.6797, \csc A = 1.778, \sec A = 1.209, \cot A = 1.471$$

Exercise:

Problem: $b = 40, c = 41$

Exercise:

Problem: $a = 84, b = 13$

Solution:

a. $c = 85$ b. $\sin A = \frac{84}{85}, \cos A = \frac{13}{85}, \tan A = \frac{84}{13}, \csc A = \frac{85}{84}, \sec A = \frac{85}{13}, \cot A = \frac{13}{84}$

Exercise:

Problem: $b = 28, c = 35$

For the following exercises, P is a point on the unit circle. a. Find the (exact) missing coordinate value of each point and b. find the values of the six trigonometric functions for the angle θ with a terminal side that passes through point P . Rationalize denominators.

Exercise:

Problem: $P\left(\frac{7}{25}, y\right), y > 0$

Solution:

a. $y = \frac{24}{25}$ b. $\sin \theta = \frac{24}{25}, \cos \theta = \frac{7}{25}, \tan \theta = \frac{24}{7}, \csc \theta = \frac{25}{24}, \sec \theta = \frac{25}{7}, \cot \theta = \frac{7}{24}$

Exercise:

Problem: $P\left(\frac{-15}{17}, y\right), y < 0$

Exercise:

Problem: $P\left(x, \frac{\sqrt{7}}{3}\right), x < 0$

Solution:

a. $x = \frac{-\sqrt{2}}{3}$ b.
 $\sin \theta = \frac{\sqrt{7}}{3}, \cos \theta = \frac{-\sqrt{2}}{3}, \tan \theta = \frac{-\sqrt{14}}{2}, \csc \theta = \frac{3\sqrt{7}}{7}, \sec \theta = \frac{-3\sqrt{2}}{2}, \cot \theta = \frac{-\sqrt{14}}{7}$

Exercise:

Problem: $P\left(x, \frac{-\sqrt{15}}{4}\right), x > 0$

For the following exercises, simplify each expression by writing it in terms of sines and cosines, then simplify. The final answer does not have to be in terms of sine and cosine only.

Exercise:

Problem: $\tan^2 x + \sin x \csc x$

Solution:

$$\sec^2 x$$

Exercise:

Problem: $\sec x \sin x \cot x$

Exercise:

Problem: $\frac{\tan^2 x}{\sec^2 x}$

Solution:

$$\sin^2 x$$

Exercise:

Problem: $\sec x - \cos x$

Exercise:

Problem: $(1 + \tan \theta)^2 - 2 \tan \theta$

Solution:

$$\sec^2 \theta$$

Exercise:

Problem: $\sin x (\csc x - \sin x)$

Exercise:

Problem: $\frac{\cos t}{\sin t} + \frac{\sin t}{1 + \cos t}$

Solution:

$$\frac{1}{\sin t} = \csc t$$

Exercise:

Problem: $\frac{1 + \tan^2 \alpha}{1 + \cot^2 \alpha}$

For the following exercises, verify that each equation is an identity.

Exercise:

Problem: $\frac{\tan\theta\cot\theta}{\csc\theta} = \sin\theta$

Exercise:

Problem: $\frac{\sec^2\theta}{\tan\theta} = \sec\theta\csc\theta$

Exercise:

Problem: $\frac{\sin t}{\csc t} + \frac{\cos t}{\sec t} = 1$

Exercise:

Problem: $\frac{\sin x}{\cos x + 1} + \frac{\cos x - 1}{\sin x} = 0$

Exercise:

Problem: $\cot\gamma + \tan\gamma = \sec\gamma\csc\gamma$

Exercise:

Problem: $\sin^2\beta + \tan^2\beta + \cos^2\beta = \sec^2\beta$

Exercise:

Problem: $\frac{1}{1-\sin\alpha} + \frac{1}{1+\sin\alpha} = 2\sec^2\alpha$

Exercise:

Problem: $\frac{\tan\theta - \cot\theta}{\sin\theta\cos\theta} = \sec^2\theta - \csc^2\theta$

For the following exercises, solve the trigonometric equations on the interval $0 \leq \theta < 2\pi$.

Exercise:

Problem: $2\sin\theta - 1 = 0$

Solution:

$$\left\{ \frac{\pi}{6}, \frac{5\pi}{6} \right\}$$

Exercise:

Problem: $1 + \cos\theta = \frac{1}{2}$

Exercise:

Problem: $2\tan^2\theta = 2$

Solution:

$$\left\{ \frac{\pi}{4}, \frac{3\pi}{4}, \frac{5\pi}{4}, \frac{7\pi}{4} \right\}$$

Exercise:

Problem: $4\sin^2\theta - 2 = 0$

Exercise:

Problem: $\sqrt{3}\cot\theta + 1 = 0$

Solution:

$$\left\{ \frac{2\pi}{3}, \frac{5\pi}{3} \right\}$$

Exercise:

Problem: $3\sec\theta - 2\sqrt{3} = 0$

Exercise:

Problem: $2\cos\theta\sin\theta = \sin\theta$

Solution:

$$\left\{ 0, \pi, \frac{\pi}{3}, \frac{5\pi}{3} \right\}$$

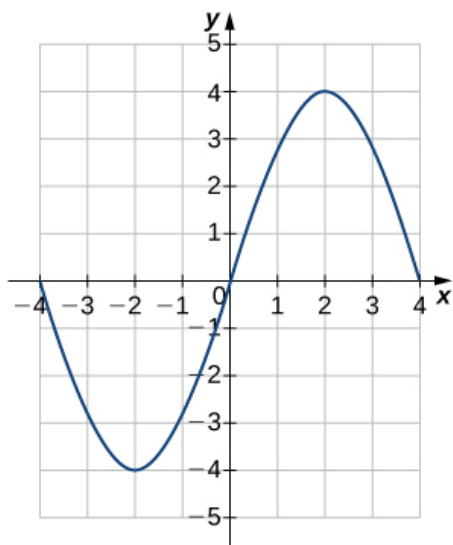
Exercise:

Problem: $\csc^2\theta + 2\csc\theta + 1 = 0$

For the following exercises, each graph is of the form $y = A\sin Bx$ or $y = A\cos Bx$, where $B > 0$. Write the equation of the graph.

Exercise:

Problem:

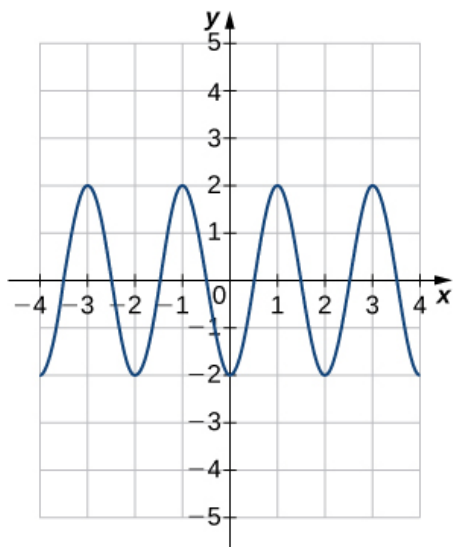


Solution:

$$y = 4 \sin \left(\frac{\pi}{4} x \right)$$

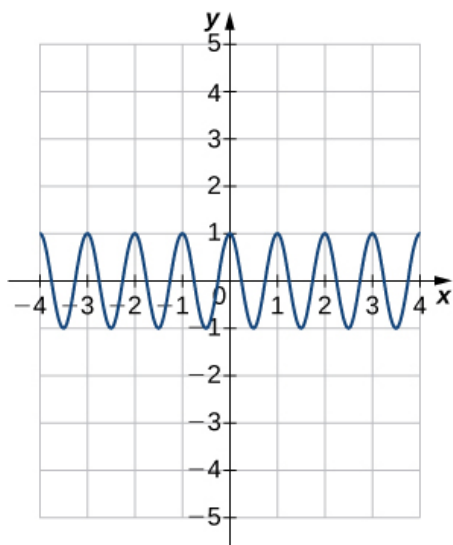
Exercise:

Problem:



Exercise:

Problem:

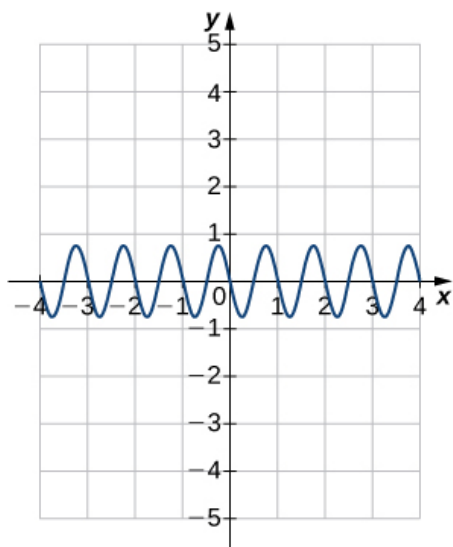


Solution:

$$y = \cos(2\pi x)$$

Exercise:

Problem:



For the following exercises, find a. the amplitude, b. the period, and c. the phase shift with direction for each function.

Exercise:

Problem: $y = \sin\left(x - \frac{\pi}{4}\right)$

Solution:

a. 1 b. 2π c. $\frac{\pi}{4}$ units to the right

Exercise:

Problem: $y = 3 \cos(2x + 3)$

Exercise:

Problem: $y = \frac{-1}{2} \sin\left(\frac{1}{4}x\right)$

Solution:

a. $\frac{1}{2}$ b. 8π c. No phase shift

Exercise:

Problem: $y = 2 \cos\left(x - \frac{\pi}{3}\right)$

Exercise:

Problem: $y = -3 \sin(\pi x + 2)$

Solution:

a. 3 b. 2 c. $\frac{2}{\pi}$ units to the left

Exercise:

Problem: $y = 4 \cos\left(2x - \frac{\pi}{2}\right)$

Exercise:

Problem:

[T] The diameter of a wheel rolling on the ground is 40 in. If the wheel rotates through an angle of 120° , how many inches does it move? Approximate to the nearest whole inch.

Solution:

Approximately 42 in.

Exercise:

Problem:

[T] Find the length of the arc intercepted by central angle θ in a circle of radius r . Round to the nearest hundredth.

a. $r = 12.8$ cm, $\theta = \frac{5\pi}{6}$ rad b. $r = 4.378$ cm, $\theta = \frac{7\pi}{6}$ rad c. $r = 0.964$ cm, $\theta = 50^\circ$ d.
 $r = 8.55$ cm, $\theta = 325^\circ$

Exercise:**Problem:**

[T] As a point P moves around a circle, the measure of the angle changes. The measure of how fast the angle is changing is called *angular speed*, ω , and is given by $\omega = \theta/t$, where θ is in radians and t is time. Find the angular speed for the given data. Round to the nearest thousandth.

- a. $\theta = \frac{7\pi}{4}$ rad, $t = 10$ sec b. $\theta = \frac{3\pi}{5}$ rad, $t = 8$ sec c. $\theta = \frac{2\pi}{9}$ rad, $t = 1$ min d.
 $\theta = 23.76$ rad, $t = 14$ min
-

Solution:

- a. 0.550 rad/sec b. 0.236 rad/sec c. 0.698 rad/min d. 1.697 rad/min

Exercise:**Problem:**

[T] A total of 250,000 m² of land is needed to build a nuclear power plant. Suppose it is decided that the area on which the power plant is to be built should be circular.

- a. Find the radius of the circular land area.
b. If the land area is to form a 45° sector of a circle instead of a whole circle, find the length of the curved side.

Exercise:

Problem: [T] The area of an isosceles triangle with equal sides of length x is

$$\frac{1}{2}x^2\sin\theta,$$

where θ is the angle formed by the two sides. Find the area of an isosceles triangle with equal sides of length 8 in. and angle $\theta = 5\pi/12$ rad.

Solution:

$$\approx 30.9 \text{ in}^2$$

Exercise:**Problem:**

[T] A particle travels in a circular path at a constant angular speed ω . The angular speed is modeled by the function $\omega = 9|\cos(\pi t - \pi/12)|$. Determine the angular speed at $t = 9$ sec.

Exercise:

Problem: [T] An alternating current for outlets in a home has voltage given by the function

$$V(t) = 150\cos 368t,$$

where V is the voltage in volts at time t in seconds.

- Find the period of the function and interpret its meaning.
- Determine the number of periods that occur when 1 sec has passed.

Solution:

a. $\pi/184$; the voltage repeats every $\pi/184$ sec b. Approximately 59 periods

Exercise:

Problem: [T] The number of hours of daylight in a northeast city is modeled by the function

Equation:

$$N(t) = 12 + 3 \sin \left[\frac{2\pi}{365}(t - 79) \right],$$

where t is the number of days after January 1.

- Find the amplitude and period.
- Determine the number of hours of daylight on the longest day of the year.
- Determine the number of hours of daylight on the shortest day of the year.
- Determine the number of hours of daylight 90 days after January 1.
- Sketch the graph of the function for one period starting on January 1.

Exercise:

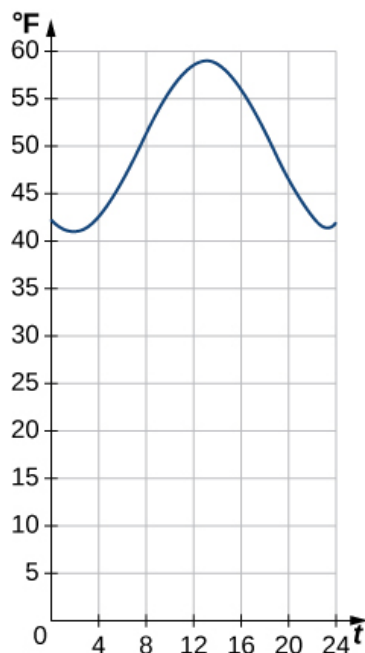
Problem:

[T] Suppose that $T = 50 + 10 \sin \left[\frac{\pi}{12}(t - 8) \right]$ is a mathematical model of the temperature (in degrees Fahrenheit) at t hours after midnight on a certain day of the week.

- Determine the amplitude and period.
- Find the temperature 7 hours after midnight.
- At what time does $T = 60^\circ$?
- Sketch the graph of T over $0 \leq t \leq 24$.

Solution:

a. Amplitude = 10; period = 24 b. $47.4^\circ F$ c. 14 hours later, or 2 p.m. d.



Exercise:

Problem:

[T] The function $H(t) = 8\sin\left(\frac{\pi}{6}t\right)$ models the height H (in feet) of the tide t hours after midnight. Assume that $t = 0$ is midnight.

- Find the amplitude and period.
- Graph the function over one period.
- What is the height of the tide at 4:30 a.m.?

Glossary

periodic function

a function is periodic if it has a repeating pattern as the values of x move from left to right

radians

for a circular arc of length s on a circle of radius 1, the radian measure of the associated angle θ is s

trigonometric functions

functions of an angle defined as ratios of the lengths of the sides of a right triangle

trigonometric identity

an equation involving trigonometric functions that is true for all angles θ for which the functions in the equation are defined

Inverse Functions

- Determine the conditions for when a function has an inverse.
- Use the horizontal line test to recognize when a function is one-to-one.
- Find the inverse of a given function.
- Draw the graph of an inverse function.
- Evaluate inverse trigonometric functions.

An inverse function reverses the operation done by a particular function. In other words, whatever a function does, the inverse function undoes it. In this section, we define an inverse function formally and state the necessary conditions for an inverse function to exist. We examine how to find an inverse function and study the relationship between the graph of a function and the graph of its inverse. Then we apply these ideas to define and discuss properties of the inverse trigonometric functions.

Existence of an Inverse Function

We begin with an example. Given a function f and an output $y = f(x)$, we are often interested in finding what value or values x were mapped to y by f . For example, consider the function $f(x) = x^3 + 4$. Since any output $y = x^3 + 4$, we can solve this equation for x to find that the input is $x = \sqrt[3]{y - 4}$. This equation defines x as a function of y . Denoting this function as f^{-1} , and writing $x = f^{-1}(y) = \sqrt[3]{y - 4}$, we see that for any x in the domain of f , $f^{-1}(f(x)) = f^{-1}(x^3 + 4) = x$. Thus, this new function, f^{-1} , “undid” what the original function f did. A function with this property is called the inverse function of the original function.

Note:

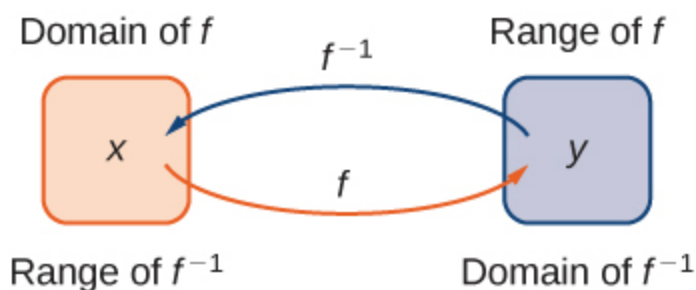
Definition

Given a function f with domain D and range R , its **inverse function** (if it exists) is the function f^{-1} with domain R and range D such that $f^{-1}(y) = x$ if $f(x) = y$. In other words, for a function f and its inverse f^{-1} ,

Equation:

$$f^{-1}(f(x)) = x \text{ for all } x \text{ in } D, \text{ and } f(f^{-1}(y)) = y \text{ for all } y \text{ in } R.$$

Note that f^{-1} is read as “f inverse.” Here, the -1 is not used as an exponent and $f^{-1}(x) \neq 1/f(x)$. [\[link\]](#) shows the relationship between the domain and range of f and the domain and range of f^{-1} .



Given a function f and its inverse f^{-1} , $f^{-1}(y) = x$ if and only if $f(x) = y$. The range of f becomes the domain of f^{-1} and the domain of f becomes the range of f^{-1} .

Recall that a function has exactly one output for each input. Therefore, to define an inverse function, we need to map each input to exactly one output. For example, let's try to find the inverse function for $f(x) = x^2$. Solving the equation $y = x^2$ for x , we arrive at the equation $x = \pm\sqrt{y}$. This equation does not describe x as a function of y because there are two solutions to this equation for every $y > 0$. The problem with trying to find an inverse function for $f(x) = x^2$ is that two inputs are sent to the same output for each output $y > 0$. The function $f(x) = x^3 + 4$ discussed earlier did not have this problem. For that function, each input was sent to a

different output. A function that sends each input to a *different* output is called a one-to-one function.

Note:

Definition

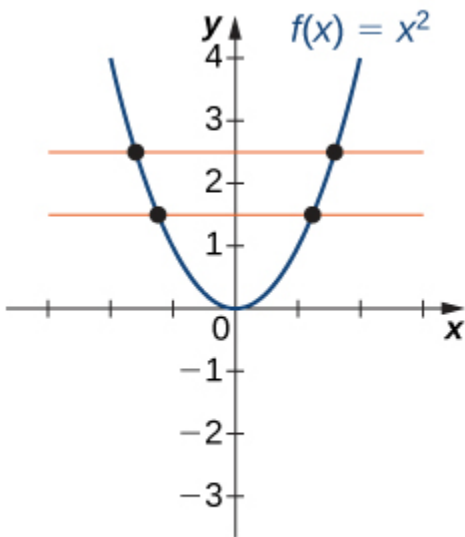
We say a f is a **one-to-one function** if $f(x_1) \neq f(x_2)$ when $x_1 \neq x_2$.

One way to determine whether a function is one-to-one is by looking at its graph. If a function is one-to-one, then no two inputs can be sent to the same output. Therefore, if we draw a horizontal line anywhere in the xy -plane, according to the **horizontal line test**, it cannot intersect the graph more than once. We note that the horizontal line test is different from the vertical line test. The vertical line test determines whether a graph is the graph of a function. The horizontal line test determines whether a function is one-to-one ([\[link\]](#)).

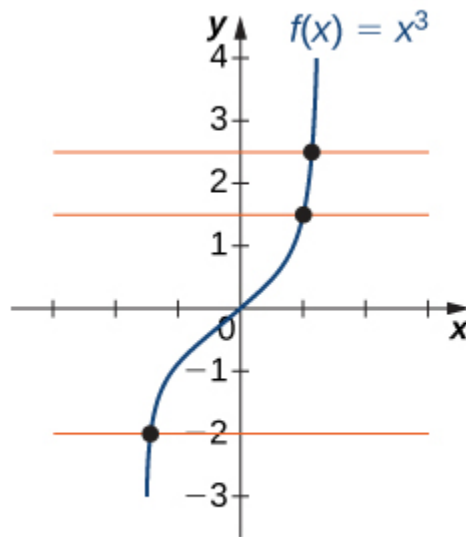
Note:

Rule: Horizontal Line Test

A function f is one-to-one if and only if every horizontal line intersects the graph of f no more than once.



(a)



(b)

- (a) The function $f(x) = x^2$ is not one-to-one because it fails the horizontal line test. (b) The function $f(x) = x^3$ is one-to-one because it passes the horizontal line test.

Example:

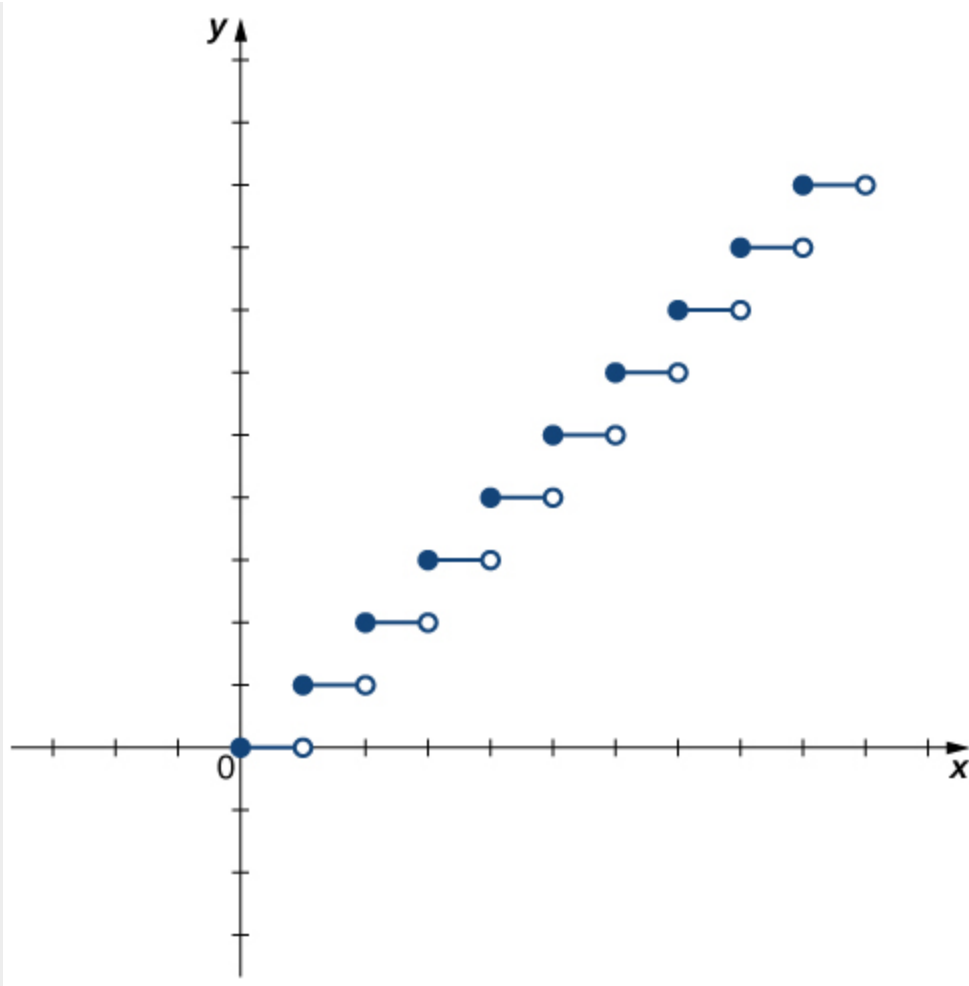
Exercise:

Problem:

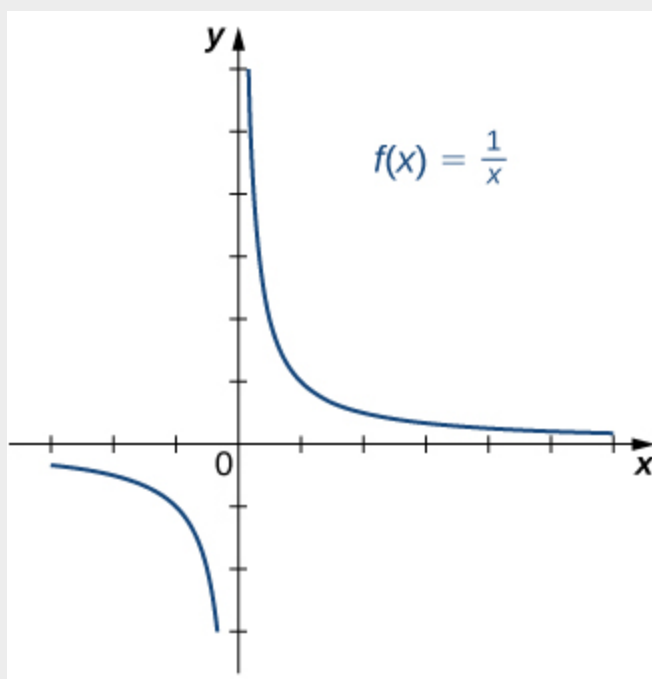
Determining Whether a Function Is One-to-One

For each of the following functions, use the horizontal line test to determine whether it is one-to-one.

a.

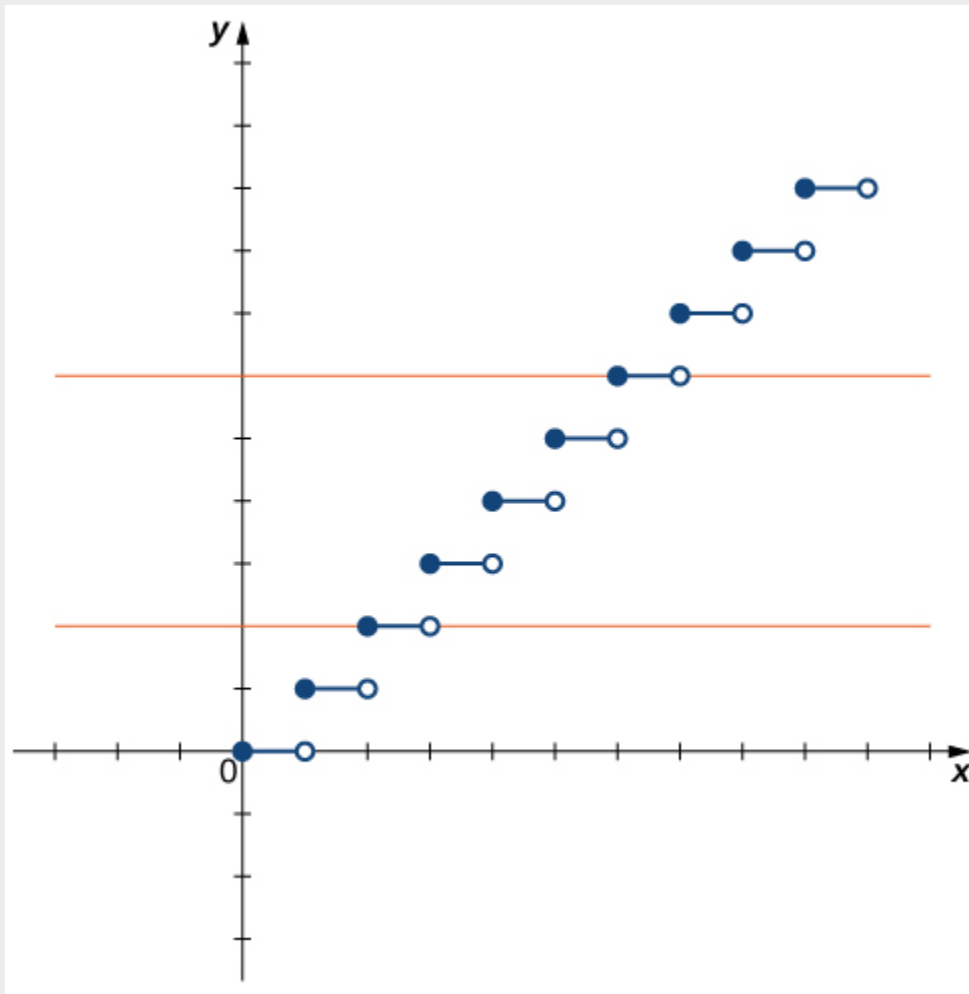


b.

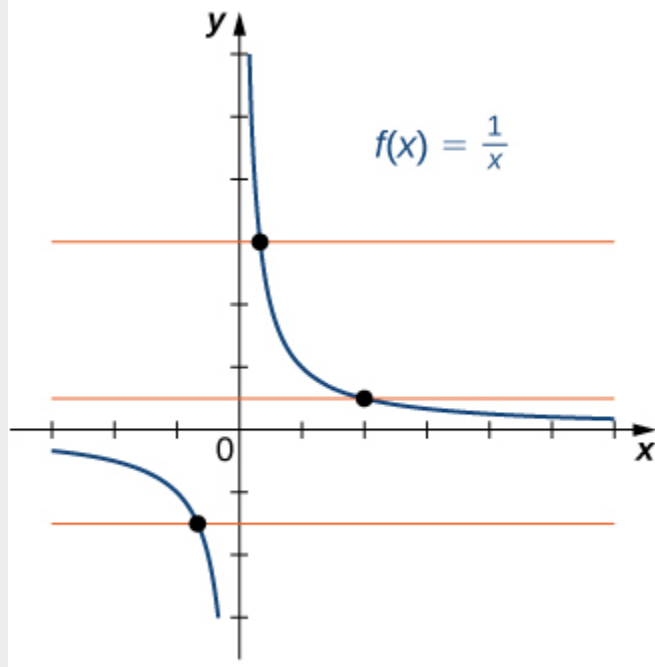


Solution:

- a. Since the horizontal line $y = n$ for any integer $n \geq 0$ intersects the graph more than once, this function is not one-to-one.



- b. Since every horizontal line intersects the graph once (at most), this function is one-to-one.

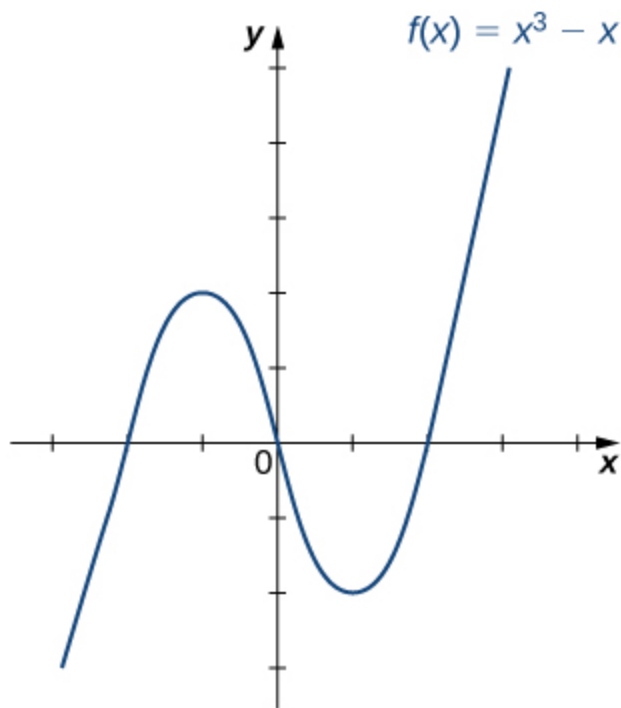


Note:

Exercise:

Problem:

Is the function f graphed in the following image one-to-one?



Solution:

No.

Hint

Use the horizontal line test.

Finding a Function's Inverse

We can now consider one-to-one functions and show how to find their inverses. Recall that a function maps elements in the domain of f to elements in the range of f . The inverse function maps each element from the range of f back to its corresponding element from the domain of f . Therefore, to find the inverse function of a one-to-one function f , given any y in the range of f , we need to determine which x in the domain of f satisfies $f(x) = y$. Since f is one-to-one, there is exactly one such value x . We can find that value x by solving the equation $f(x) = y$ for x . Doing so,

we are able to write x as a function of y where the domain of this function is the range of f and the range of this new function is the domain of f . Consequently, this function is the inverse of f , and we write $x = f^{-1}(y)$. Since we typically use the variable x to denote the independent variable and y to denote the dependent variable, we often interchange the roles of x and y , and write $y = f^{-1}(x)$. Representing the inverse function in this way is also helpful later when we graph a function f and its inverse f^{-1} on the same axes.

Note:**Problem-Solving Strategy: Finding an Inverse Function**

1. Solve the equation $y = f(x)$ for x .
2. Interchange the variables x and y and write $y = f^{-1}(x)$.

Example:**Exercise:****Problem:****Finding an Inverse Function**

Find the inverse for the function $f(x) = 3x - 4$. State the domain and range of the inverse function. Verify that $f^{-1}(f(x)) = x$.

Solution:

Follow the steps outlined in the strategy.

Step 1. If $y = 3x - 4$, then $3x = y + 4$ and $x = \frac{1}{3}y + \frac{4}{3}$.

Step 2. Rewrite as $y = \frac{1}{3}x + \frac{4}{3}$ and let $y = f^{-1}(x)$.

Therefore, $f^{-1}(x) = \frac{1}{3}x + \frac{4}{3}$.

Since the domain of f is $(-\infty, \infty)$, the range of f^{-1} is $(-\infty, \infty)$.
Since the range of f is $(-\infty, \infty)$, the domain of f^{-1} is $(-\infty, \infty)$.

You can verify that $f^{-1}(f(x)) = x$ by writing

Equation:

$$f^{-1}(f(x)) = f^{-1}(3x - 4) = \frac{1}{3}(3x - 4) + \frac{4}{3} = x - \frac{4}{3} + \frac{4}{3} = x.$$

Note that for $f^{-1}(x)$ to be the inverse of $f(x)$, both $f^{-1}(f(x)) = x$ and $f(f^{-1}(x)) = x$ for all x in the domain of the inside function.

Note:

Exercise:

Problem:

Find the inverse of the function $f(x) = 3x/(x - 2)$. State the domain and range of the inverse function.

Solution:

$f^{-1}(x) = \frac{2x}{x-3}$. The domain of f^{-1} is $\{x|x \neq 3\}$. The range of f^{-1} is $\{y|y \neq 2\}$.

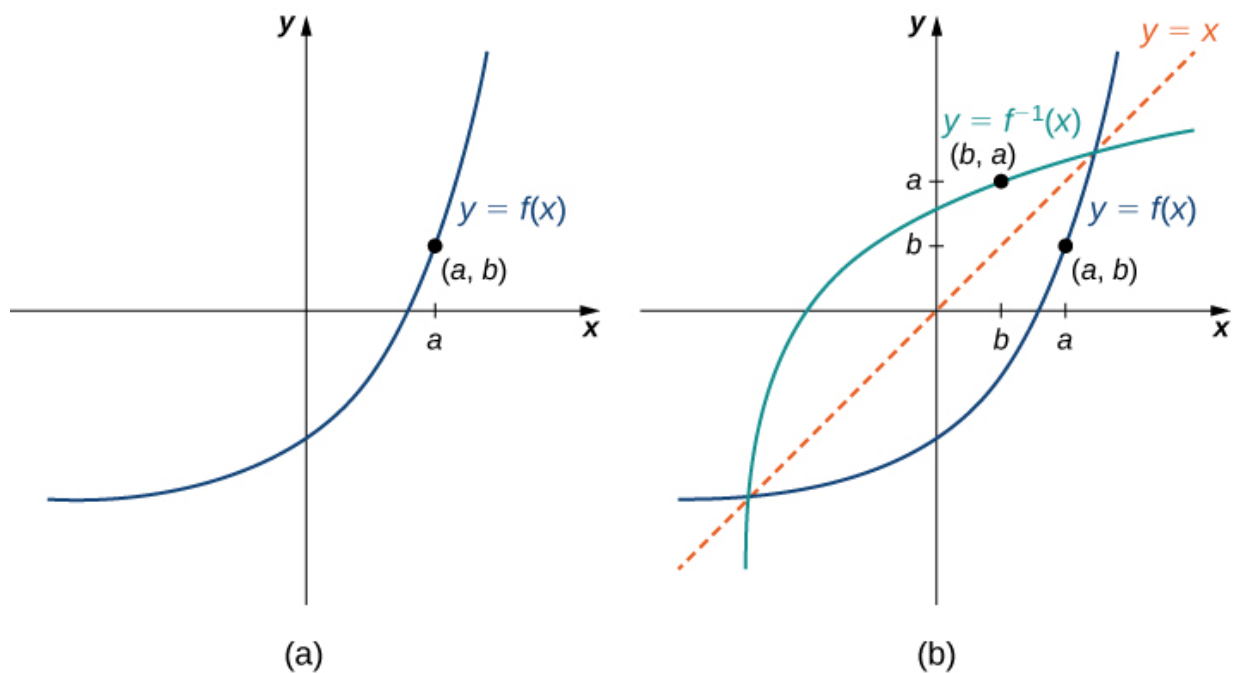
Hint

Use the [\[link\]](#) for finding inverse functions.

Graphing Inverse Functions

Let's consider the relationship between the graph of a function f and the graph of its inverse. Consider the graph of f shown in [\[link\]](#) and a point

(a, b) on the graph. Since $b = f(a)$, then $f^{-1}(b) = a$. Therefore, when we graph f^{-1} , the point (b, a) is on the graph. As a result, the graph of f^{-1} is a reflection of the graph of f about the line $y = x$.



- (a) The graph of this function f shows point (a, b) on the graph of f .
 (b) Since (a, b) is on the graph of f , the point (b, a) is on the graph of f^{-1} . The graph of f^{-1} is a reflection of the graph of f about the line $y = x$.

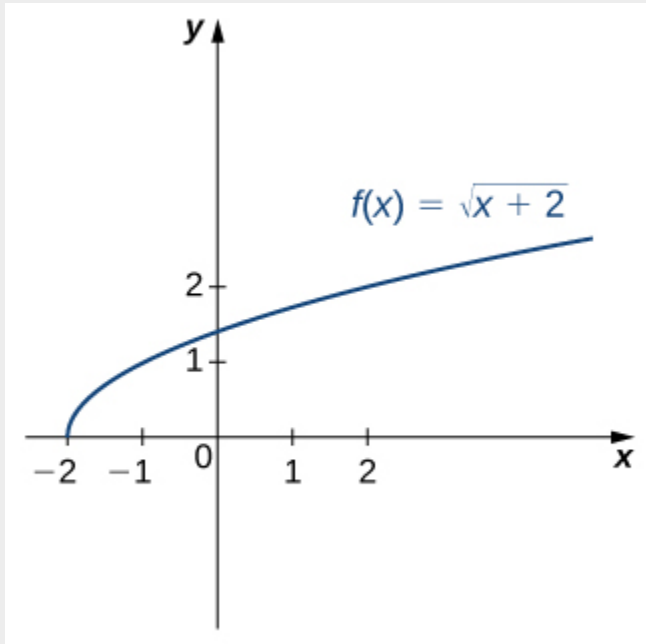
Example:

Exercise:

Problem:

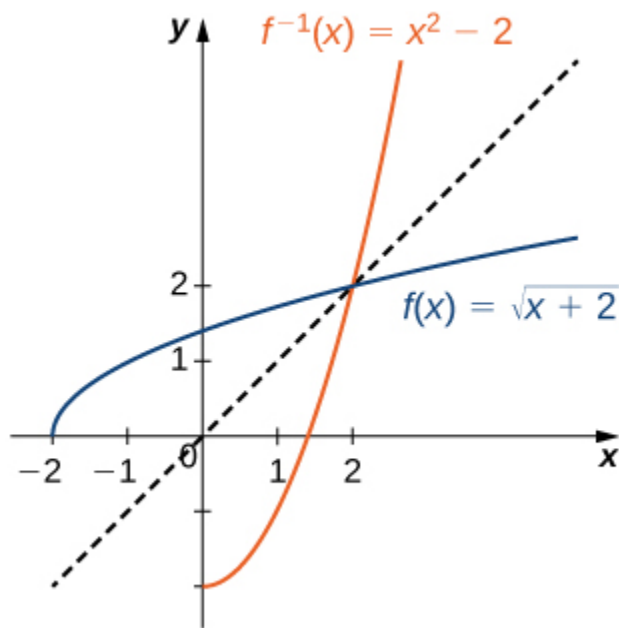
Sketching Graphs of Inverse Functions

For the graph of f in the following image, sketch a graph of f^{-1} by sketching the line $y = x$ and using symmetry. Identify the domain and range of f^{-1} .



Solution:

Reflect the graph about the line $y = x$. The domain of f^{-1} is $[0, \infty)$. The range of f^{-1} is $[-2, \infty)$. By using the preceding strategy for finding inverse functions, we can verify that the inverse function is $f^{-1}(x) = x^2 - 2$, as shown in the graph.



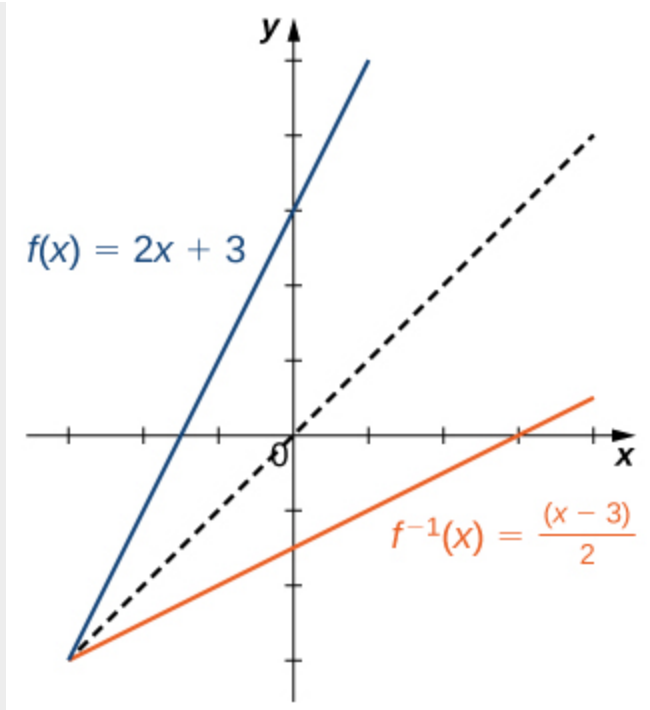
Note:

Exercise:

Problem:

Sketch the graph of $f(x) = 2x + 3$ and the graph of its inverse using the symmetry property of inverse functions.

Solution:



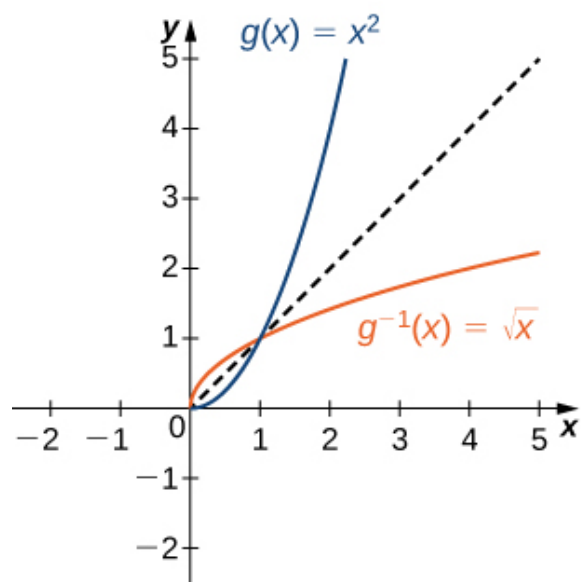
Hint

The graphs are symmetric about the line $y = x$.

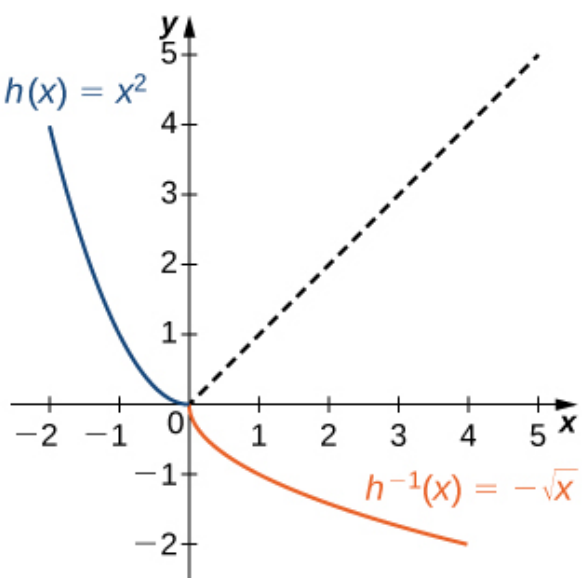
Restricting Domains

As we have seen, $f(x) = x^2$ does not have an inverse function because it is not one-to-one. However, we can choose a subset of the domain of f such that the function is one-to-one. This subset is called a **restricted domain**. By restricting the domain of f , we can define a new function g such that the domain of g is the restricted domain of f and $g(x) = f(x)$ for all x in the domain of g . Then we can define an inverse function for g on that domain. For example, since $f(x) = x^2$ is one-to-one on the interval $[0, \infty)$, we can define a new function g such that the domain of g is $[0, \infty)$ and $g(x) = x^2$ for all x in its domain. Since g is a one-to-one function, it has an inverse function, given by the formula $g^{-1}(x) = \sqrt{x}$. On the other hand, the function $f(x) = x^2$ is also one-to-one on the domain $(-\infty, 0]$. Therefore, we could also define a new function h such that the domain of h is $(-\infty, 0]$

and $h(x) = x^2$ for all x in the domain of h . Then h is a one-to-one function and must also have an inverse. Its inverse is given by the formula $h^{-1}(x) = -\sqrt{x}$ ([link](#)).



(a)



(b)

(a) For $g(x) = x^2$ restricted to $[0, \infty)$, $g^{-1}(x) = \sqrt{x}$. (b) For $h(x) = x^2$ restricted to $(-\infty, 0]$, $h^{-1}(x) = -\sqrt{x}$.

Example:

Exercise:

Problem:

Restricting the Domain

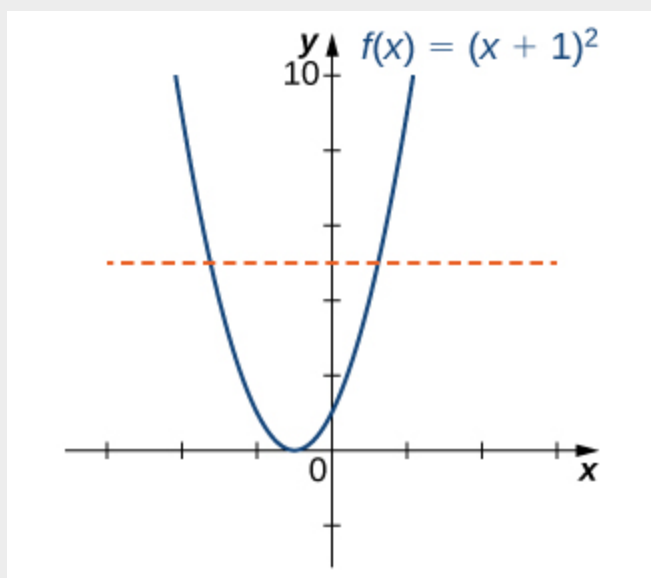
Consider the function $f(x) = (x + 1)^2$.

- Sketch the graph of f and use the horizontal line test to show that f is not one-to-one.

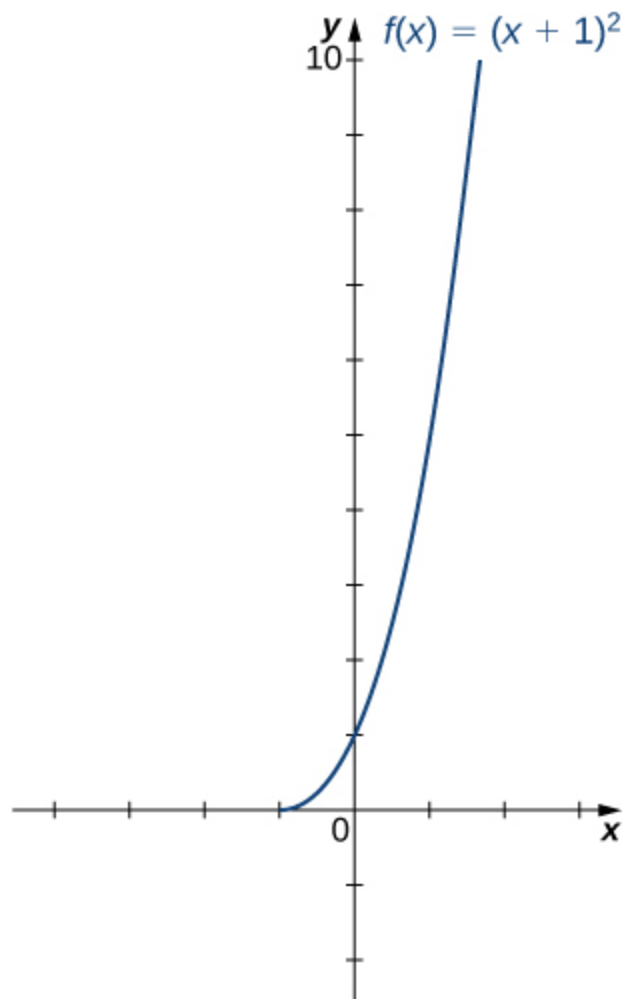
- b. Show that f is one-to-one on the restricted domain $[-1, \infty)$. Determine the domain and range for the inverse of f on this restricted domain and find a formula for f^{-1} .

Solution:

- a. The graph of f is the graph of $y = x^2$ shifted left 1 unit. Since there exists a horizontal line intersecting the graph more than once, f is not one-to-one.



- b. On the interval $[-1, \infty)$, f is one-to-one.



The domain and range of f^{-1} are given by the range and domain of f , respectively. Therefore, the domain of f^{-1} is $[0, \infty)$ and the range of f^{-1} is $[-1, \infty)$. To find a formula for f^{-1} , solve the equation $y = (x + 1)^2$ for x . If $y = (x + 1)^2$, then $x = -1 \pm \sqrt{y}$. Since we are restricting the domain to the interval where $x \geq -1$, we need $\pm\sqrt{y} \geq 0$. Therefore, $x = -1 + \sqrt{y}$. Interchanging x and y , we write $y = -1 + \sqrt{x}$ and conclude that $f^{-1}(x) = -1 + \sqrt{x}$.

Note:

Exercise:

Problem:

Consider $f(x) = 1/x^2$ restricted to the domain $(-\infty, 0)$. Verify that f is one-to-one on this domain. Determine the domain and range of the inverse of f and find a formula for f^{-1} .

Solution:

The domain of f^{-1} is $(0, \infty)$. The range of f^{-1} is $(-\infty, 0)$. The inverse function is given by the formula $f^{-1}(x) = -1/\sqrt{x}$.

Hint

The domain and range of f^{-1} is given by the range and domain of f , respectively. To find f^{-1} , solve $y = 1/x^2$ for x .

Inverse Trigonometric Functions

The six basic trigonometric functions are periodic, and therefore they are not one-to-one. However, if we restrict the domain of a trigonometric function to an interval where it is one-to-one, we can define its inverse. Consider the sine function ([\[link\]](#)). The sine function is one-to-one on an infinite number of intervals, but the standard convention is to restrict the domain to the interval $[-\frac{\pi}{2}, \frac{\pi}{2}]$. By doing so, we define the inverse sine function on the domain $[-1, 1]$ such that for any x in the interval $[-1, 1]$, the inverse sine function tells us which angle θ in the interval $[-\frac{\pi}{2}, \frac{\pi}{2}]$ satisfies $\sin \theta = x$. Similarly, we can restrict the domains of the other trigonometric functions to define **inverse trigonometric functions**, which are functions that tell us which angle in a certain interval has a specified trigonometric value.

Note:**Definition**

The inverse sine function, denoted $\sin^{-1}$ or $\arcsin$, and the inverse cosine function, denoted $\cos^{-1}$ or $\arccos$, are defined on the domain

$D = \{x \mid -1 \leq x \leq 1\}$ as follows:

Equation:

$$\sin^{-1}(x) = y \text{ if and only if } \sin(y) = x \text{ and } -\frac{\pi}{2} \leq y \leq \frac{\pi}{2};$$

$$\cos^{-1}(x) = y \text{ if and only if } \cos(y) = x \text{ and } 0 \leq y \leq \pi.$$

The inverse tangent function, denoted $\tan^{-1}$ or $\arctan$, and inverse cotangent function, denoted $\cot^{-1}$ or arccot , are defined on the domain

$D = \{x \mid -\infty < x < \infty\}$ as follows:

Equation:

$$\tan^{-1}(x) = y \text{ if and only if } \tan(y) = x \text{ and } -\frac{\pi}{2} < y < \frac{\pi}{2};$$

$$\cot^{-1}(x) = y \text{ if and only if } \cot(y) = x \text{ and } 0 < y < \pi.$$

The inverse cosecant function, denoted $\csc^{-1}$ or arccsc , and inverse secant function, denoted $\sec^{-1}$ or arcsec , are defined on the domain

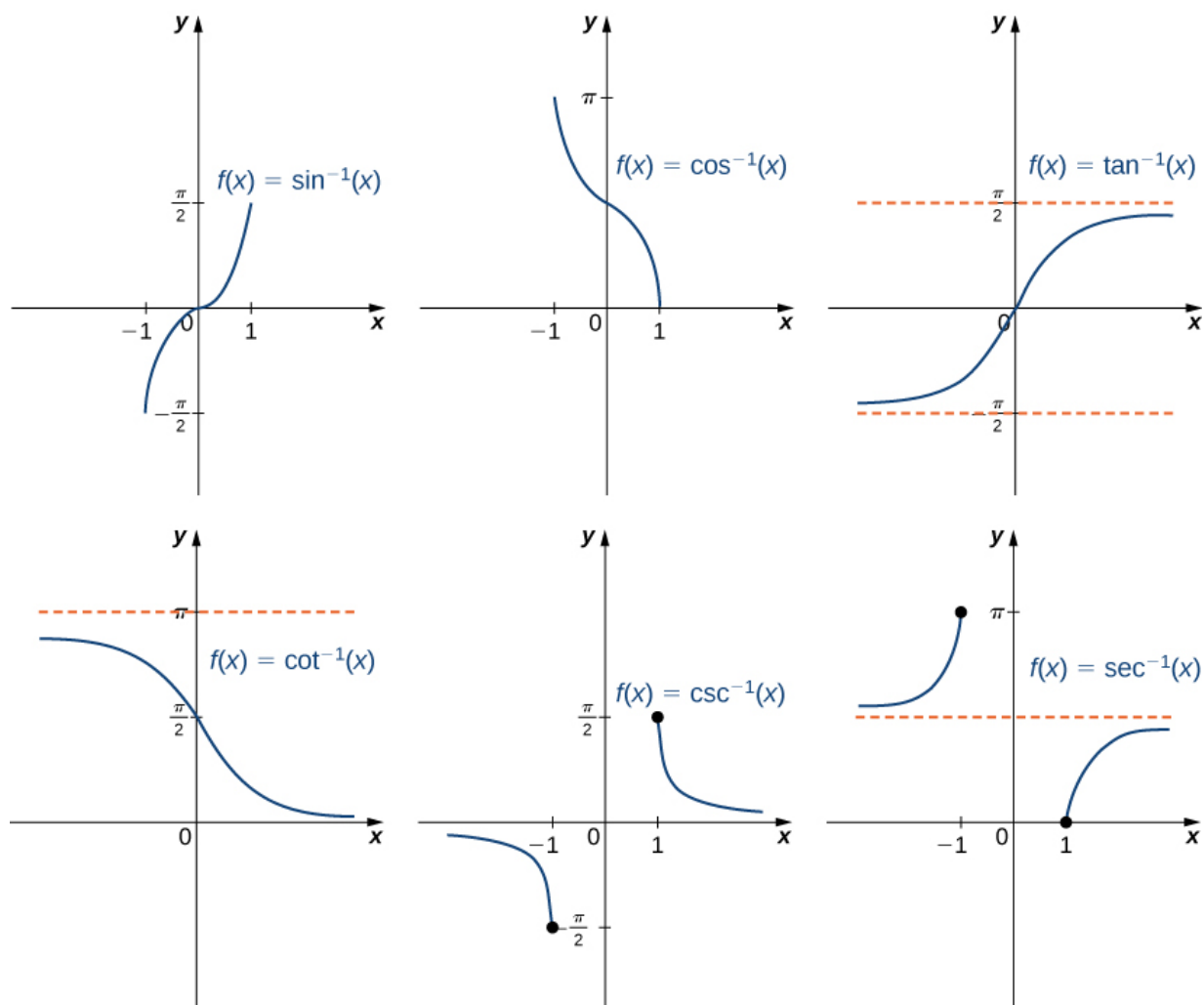
$D = \{x \mid |x| \geq 1\}$ as follows:

Equation:

$$\csc^{-1}(x) = y \text{ if and only if } \csc(y) = x \text{ and } -\frac{\pi}{2} \leq y \leq \frac{\pi}{2}, y \neq 0;$$

$$\sec^{-1}(x) = y \text{ if and only if } \sec(y) = x \text{ and } 0 \leq y \leq \pi, y \neq \pi/2.$$

To graph the inverse trigonometric functions, we use the graphs of the trigonometric functions restricted to the domains defined earlier and reflect the graphs about the line $y = x$ ([link](#)).



The graph of each of the inverse trigonometric functions is a reflection about the line $y = x$ of the corresponding restricted trigonometric function.

Note:

Go to the [following site](#) for more comparisons of functions and their inverses.

When evaluating an inverse trigonometric function, the output is an angle. For example, to evaluate $\cos^{-1}\left(\frac{1}{2}\right)$, we need to find an angle θ such that $\cos\theta = \frac{1}{2}$. Clearly, many angles have this property. However, given the definition of $\cos^{-1}$, we need the angle θ that not only solves this equation, but also lies in the interval $[0, \pi]$. We conclude that $\cos^{-1}\left(\frac{1}{2}\right) = \frac{\pi}{3}$.

We now consider a composition of a trigonometric function and its inverse. For example, consider the two expressions $\sin\left(\sin^{-1}\left(\frac{\sqrt{2}}{2}\right)\right)$ and $\sin^{-1}(\sin(\pi))$. For the first one, we simplify as follows:

Equation:

$$\sin\left(\sin^{-1}\left(\frac{\sqrt{2}}{2}\right)\right) = \sin\left(\frac{\pi}{4}\right) = \frac{\sqrt{2}}{2}.$$

For the second one, we have

Equation:

$$\sin^{-1}(\sin(\pi)) = \sin^{-1}(0) = 0.$$

The inverse function is supposed to “undo” the original function, so why isn’t $\sin^{-1}(\sin(\pi)) = \pi$? Recalling our definition of inverse functions, a function f and its inverse f^{-1} satisfy the conditions $f(f^{-1}(y)) = y$ for all y in the domain of f^{-1} and $f^{-1}(f(x)) = x$ for all x in the domain of f , so what happened here? The issue is that the inverse sine function, $\sin^{-1}$, is the inverse of the *restricted* sine function defined on the domain $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$. Therefore, for x in the interval $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$, it is true that $\sin^{-1}(\sin x) = x$. However, for values of x outside this interval, the equation does not hold, even though $\sin^{-1}(\sin x)$ is defined for all real numbers x .

What about $\sin(\sin^{-1}y)$? Does that have a similar issue? The answer is *no*. Since the domain of $\sin^{-1}$ is the interval $[-1, 1]$, we conclude that $\sin(\sin^{-1}y) = y$ if $-1 \leq y \leq 1$ and the expression is not defined for other values of y . To summarize,

Equation:

$$\sin(\sin^{-1}y) = y \text{ if } -1 \leq y \leq 1$$

and

Equation:

$$\sin^{-1}(\sin x) = x \text{ if } -\frac{\pi}{2} \leq x \leq \frac{\pi}{2}.$$

Similarly, for the cosine function,

Equation:

$$\cos(\cos^{-1}y) = y \text{ if } -1 \leq y \leq 1$$

and

Equation:

$$\cos^{-1}(\cos x) = x \text{ if } 0 \leq x \leq \pi.$$

Similar properties hold for the other trigonometric functions and their inverses.

Example:

Exercise:

Problem:

Evaluating Expressions Involving Inverse Trigonometric Functions

Evaluate each of the following expressions.

a. $\sin^{-1}\left(-\frac{\sqrt{3}}{2}\right)$

- b. $\tan \left(\tan^{-1} \left(-\frac{1}{\sqrt{3}} \right) \right)$
- c. $\cos^{-1} \left(\cos \left(\frac{5\pi}{4} \right) \right)$
- d. $\sin^{-1} \left(\cos \left(\frac{2\pi}{3} \right) \right)$

Solution:

- a. Evaluating $\sin^{-1} \left(-\sqrt{3}/2 \right)$ is equivalent to finding the angle θ such that $\sin \theta = -\sqrt{3}/2$ and $-\pi/2 \leq \theta \leq \pi/2$. The angle $\theta = -\pi/3$ satisfies these two conditions. Therefore,

$$\sin^{-1} \left(-\sqrt{3}/2 \right) = -\pi/3.$$
- b. First we use the fact that $\tan^{-1} \left(-1/\sqrt{3} \right) = -\pi/6$. Then

$$\tan \left(\tan^{-1} \left(-1/\sqrt{3} \right) \right) = -1/\sqrt{3}.$$
Therefore,

$$\tan \left(\tan^{-1} \left(-1/\sqrt{3} \right) \right) = -1/\sqrt{3}.$$
- c. To evaluate $\cos^{-1} \left(\cos \left(5\pi/4 \right) \right)$, first use the fact that $\cos \left(5\pi/4 \right) = -\sqrt{2}/2$. Then we need to find the angle θ such that $\cos \left(\theta \right) = -\sqrt{2}/2$ and $0 \leq \theta \leq \pi$. Since $3\pi/4$ satisfies both these conditions, we have

$$\cos \left(\cos^{-1} \left(5\pi/4 \right) \right) = \cos \left(\cos^{-1} \left(-\sqrt{2}/2 \right) \right) = 3\pi/4.$$
- d. Since $\cos \left(2\pi/3 \right) = -1/2$, we need to evaluate $\sin^{-1} \left(-1/2 \right)$. That is, we need to find the angle θ such that $\sin \left(\theta \right) = -1/2$ and $-\pi/2 \leq \theta \leq \pi/2$. Since $-\pi/6$ satisfies both these conditions, we can conclude that

$$\sin^{-1} \left(\cos \left(2\pi/3 \right) \right) = \sin^{-1} \left(-1/2 \right) = -\pi/6.$$

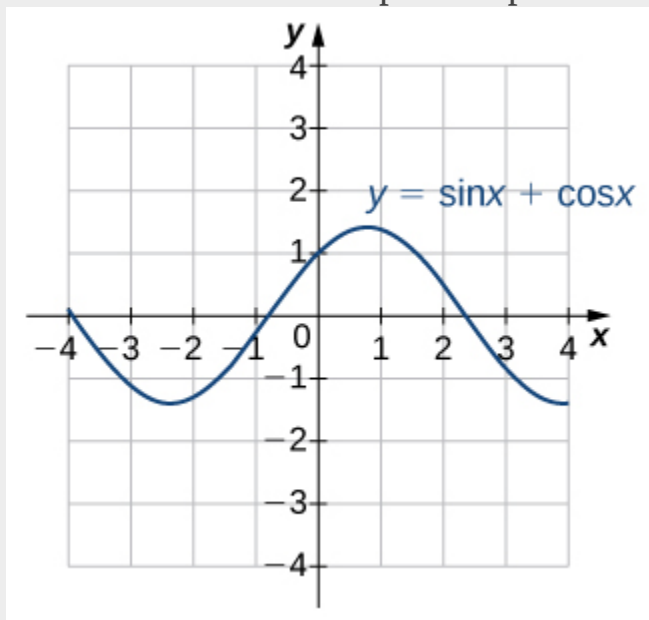
Note:

The Maximum Value of a Function

In many areas of science, engineering, and mathematics, it is useful to know the maximum value a function can obtain, even if we don't know its exact value at a given instant. For instance, if we have a function describing the strength of a roof beam, we would want to know the maximum weight the beam can support without breaking. If we have a function that describes the speed of a train, we would want to know its maximum speed before it jumps off the rails. Safe design often depends on knowing maximum values.

This project describes a simple example of a function with a maximum value that depends on two equation coefficients. We will see that maximum values can depend on several factors other than the independent variable x .

1. Consider the graph in [\[link\]](#) of the function $y = \sin x + \cos x$. Describe its overall shape. Is it periodic? How do you know?



The graph of $y = \sin x + \cos x$.

Using a graphing calculator or other graphing device, estimate the x - and y -values of the maximum point for the graph (the first such point where $x > 0$). It may be helpful to express the x -value as a multiple of π .

2. Now consider other graphs of the form $y = A\sin x + B\cos x$ for various values of A and B . Sketch the graph when $A = 2$ and $B = 1$, and find the x - and y -values for the maximum point. (Remember to express the x -value as a multiple of π , if possible.) Has it moved?
3. Repeat for $A = 1, B = 2$. Is there any relationship to what you found in part (2)?
4. Complete the following table, adding a few choices of your own for A and B :

A	B	x	y		A	B	x	y
0	1				$\sqrt{3}$	1		
1	0				1	$\sqrt{3}$		
1	1				12	5		
1	2				5	12		
2	1							
2	2							
3	4							
4	3							

5. Try to figure out the formula for the y -values.
6. The formula for the x -values is a little harder. The most helpful points from the table are $(1, 1)$, $(1, \sqrt{3})$, $(\sqrt{3}, 1)$. (*Hint: Consider inverse trigonometric functions.*)

7. If you found formulas for parts (5) and (6), show that they work together. That is, substitute the x -value formula you found into $y = A \sin x + B \cos x$ and simplify it to arrive at the y -value formula you found.

Key Concepts

- For a function to have an inverse, the function must be one-to-one. Given the graph of a function, we can determine whether the function is one-to-one by using the horizontal line test.
- If a function is not one-to-one, we can restrict the domain to a smaller domain where the function is one-to-one and then define the inverse of the function on the smaller domain.
- For a function f and its inverse f^{-1} , $f(f^{-1}(x)) = x$ for all x in the domain of f^{-1} and $f^{-1}(f(x)) = x$ for all x in the domain of f .
- Since the trigonometric functions are periodic, we need to restrict their domains to define the inverse trigonometric functions.
- The graph of a function f and its inverse f^{-1} are symmetric about the line $y = x$.

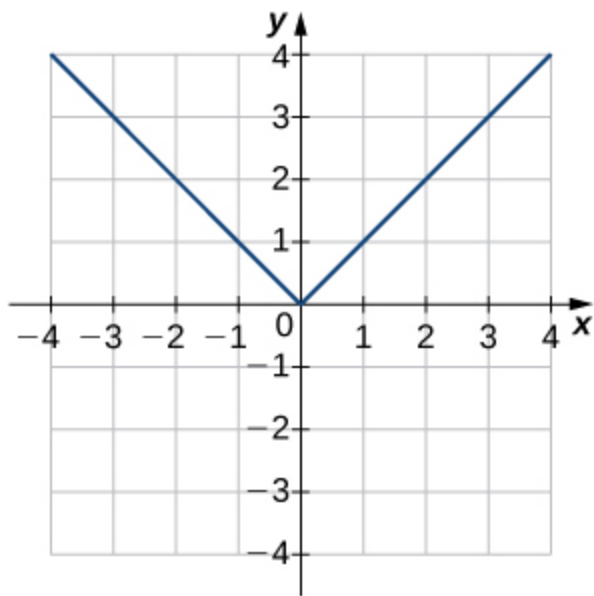
Key Equations

- **Inverse functions**
 $f^{-1}(f(x)) = x$ for all x in D , and $f(f^{-1}(y)) = y$ for all y in R .

For the following exercises, use the horizontal line test to determine whether each of the given graphs is one-to-one.

Exercise:

Problem:

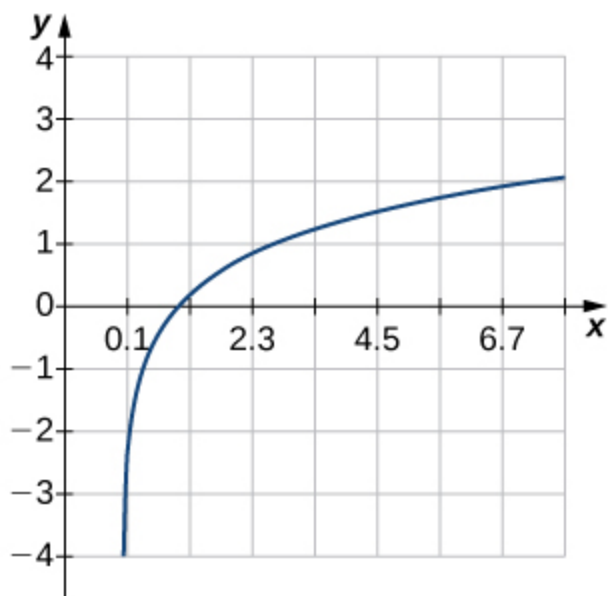


Solution:

Not one-to-one

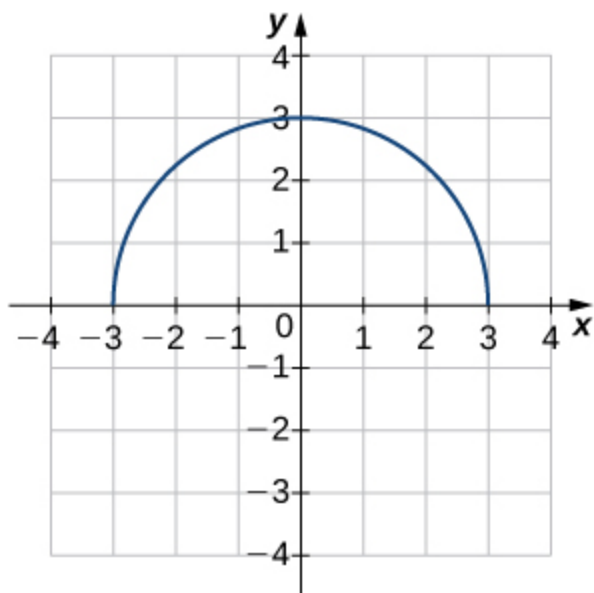
Exercise:

Problem:



Exercise:

Problem:

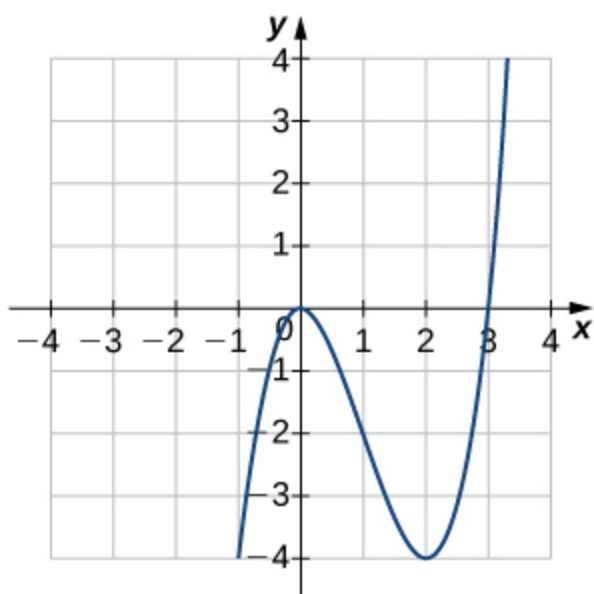


Solution:

Not one-to-one

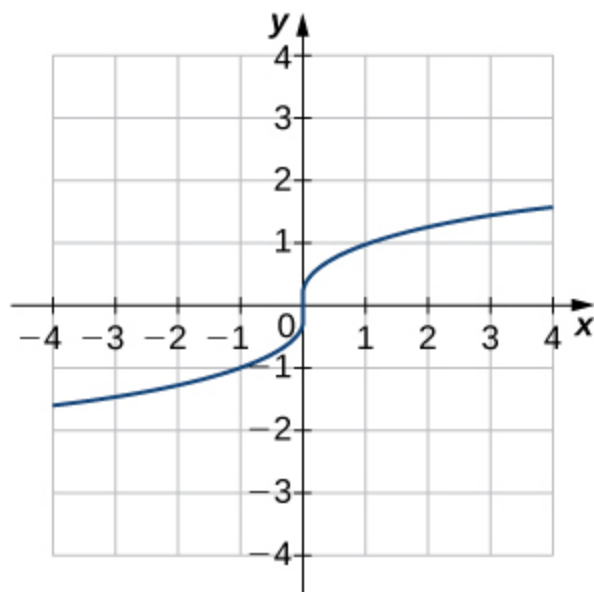
Exercise:

Problem:



Exercise:

Problem:

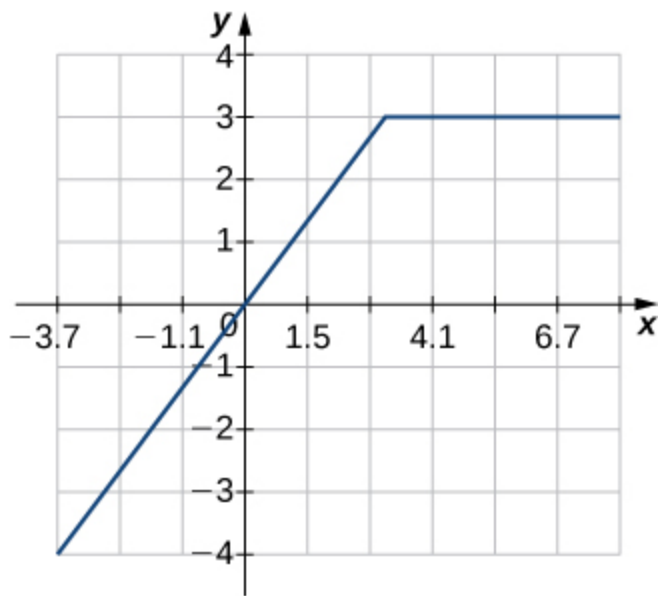


Solution:

One-to-one

Exercise:

Problem:



For the following exercises, a. find the inverse function, and b. find the domain and range of the inverse function.

Exercise:

Problem: $f(x) = x^2 - 4, x \geq 0$

Solution:

a. $f^{-1}(x) = \sqrt{x + 4}$ b. Domain : $x \geq -4$, range: $y \geq 0$

Exercise:

Problem: $f(x) = \sqrt[3]{x - 4}$

Exercise:

Problem: $f(x) = x^3 + 1$

Solution:

a. $f^{-1}(x) = \sqrt[3]{x-1}$ b. Domain: all real numbers, range: all real numbers

Exercise:

Problem: $f(x) = (x-1)^2, x \leq 1$

Exercise:

Problem: $f(x) = \sqrt{x-1}$

Solution:

a. $f^{-1}(x) = x^2 + 1$, b. Domain: $x \geq 0$, range: $y \geq 1$

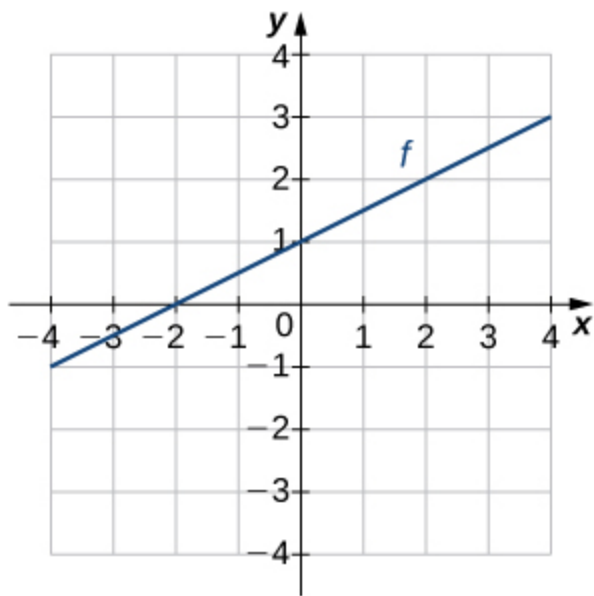
Exercise:

Problem: $f(x) = \frac{1}{x+2}$

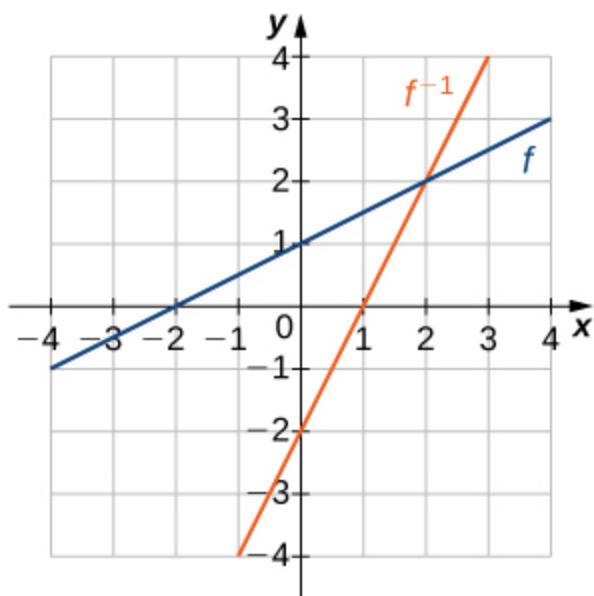
For the following exercises, use the graph of f to sketch the graph of its inverse function.

Exercise:

Problem:

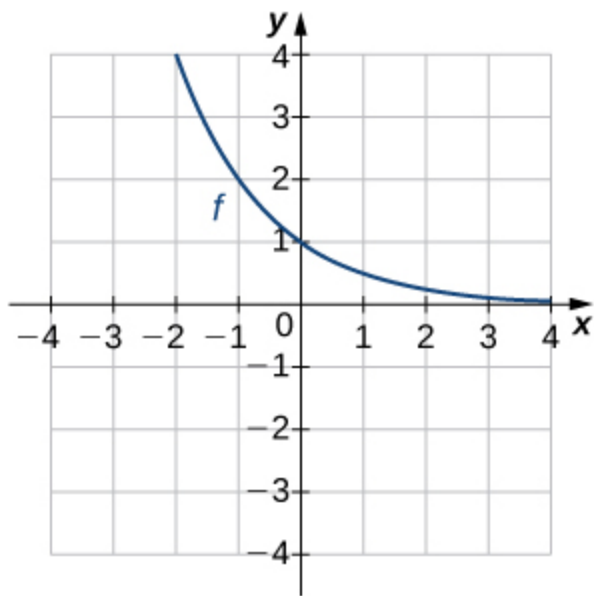


Solution:



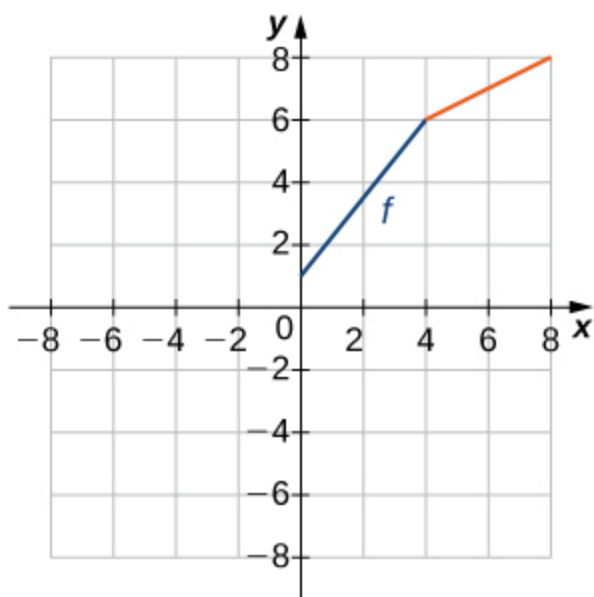
Exercise:

Problem:

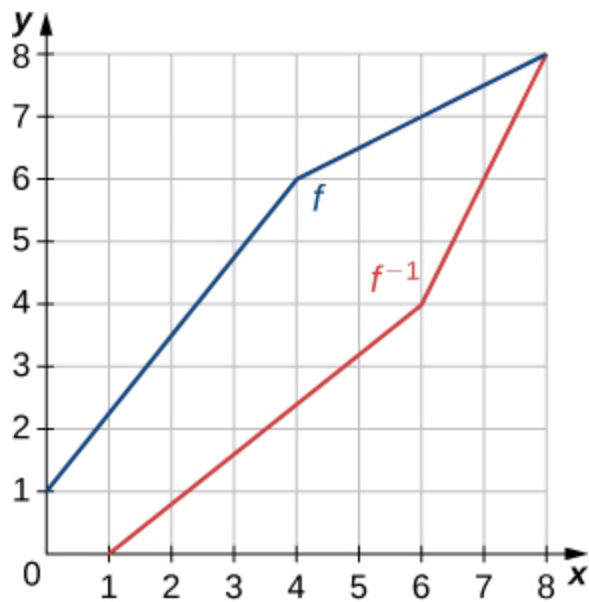


Exercise:

Problem:

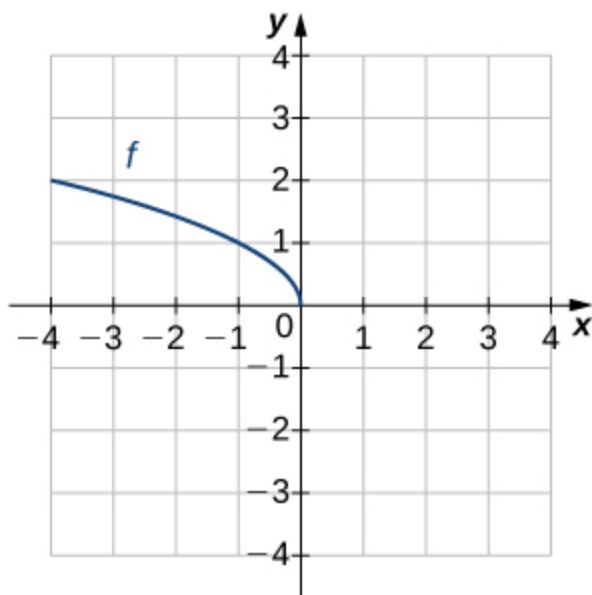


Solution:



Exercise:

Problem:



For the following exercises, use composition to determine which pairs of functions are inverses.

Exercise:

Problem: $f(x) = 8x, g(x) = \frac{x}{8}$

Solution:

These are inverses.

Exercise:

Problem: $f(x) = 8x + 3, g(x) = \frac{x-3}{8}$

Exercise:

Problem: $f(x) = 5x - 7, g(x) = \frac{x+5}{7}$

Solution:

These are not inverses.

Exercise:

Problem: $f(x) = \frac{2}{3}x + 2, g(x) = \frac{3}{2}x + 3$

Exercise:

Problem: $f(x) = \frac{1}{x-1}, x \neq 1, g(x) = \frac{1}{x} + 1, x \neq 0$

Solution:

These are inverses.

Exercise:

Problem: $f(x) = x^3 + 1, g(x) = (x - 1)^{1/3}$

Exercise:

Problem: $f(x) = x^2 + 2x + 1, x \geq -1, \quad g(x) = -1 + \sqrt{x}, x \geq 0$

Solution:

These are inverses.

Exercise:

Problem:

$$f(x) = \sqrt{4 - x^2}, 0 \leq x \leq 2, g(x) = \sqrt{4 - x^2}, 0 \leq x \leq 2$$

For the following exercises, evaluate the functions. Give the exact value.

Exercise:

Problem: $\tan^{-1}\left(\frac{\sqrt{3}}{3}\right)$

Solution:

$$\frac{\pi}{6}$$

Exercise:

Problem: $\cos^{-1}\left(-\frac{\sqrt{2}}{2}\right)$

Exercise:

Problem: $\cot^{-1}(1)$

Solution:

$$\frac{\pi}{4}$$

Exercise:

Problem: $\sin^{-1}(-1)$

Exercise:

Problem: $\cos^{-1}\left(\frac{\sqrt{3}}{2}\right)$

Solution:

$$\frac{\pi}{6}$$

Exercise:

Problem: $\cos\left(\tan^{-1}\left(\sqrt{3}\right)\right)$

Exercise:

Problem: $\sin\left(\cos^{-1}\left(\frac{\sqrt{2}}{2}\right)\right)$

Solution:

$$\frac{\sqrt{2}}{2}$$

Exercise:

Problem: $\sin^{-1}\left(\sin\left(\frac{\pi}{3}\right)\right)$

Exercise:

Problem: $\tan^{-1}\left(\tan\left(-\frac{\pi}{6}\right)\right)$

Solution:

$$-\frac{\pi}{6}$$

Exercise:

Problem:

The function $C = T(F) = (5/9)(F - 32)$ converts degrees Fahrenheit to degrees Celsius.

- Find the inverse function $F = T^{-1}(C)$
- What is the inverse function used for?

Exercise:**Problem:**

[T] The velocity V (in centimeters per second) of blood in an artery at a distance x cm from the center of the artery can be modeled by the function $V = f(x) = 500(0.04 - x^2)$ for $0 \leq x \leq 0.2$.

- Find $x = f^{-1}(V)$.
- Interpret what the inverse function is used for.
- Find the distance from the center of an artery with a velocity of 15 cm/sec, 10 cm/sec, and 5 cm/sec.

Solution:

- a. $x = f^{-1}(V) = \sqrt{0.04 - \frac{V}{500}}$ b. The inverse function determines the distance from the center of the artery at which blood is flowing with velocity V . c. 0.1 cm; 0.14 cm; 0.17 cm

Exercise:**Problem:**

A function that converts dress sizes in the United States to those in Europe is given by $D(x) = 2x + 24$.

- Find the European dress sizes that correspond to sizes 6, 8, 10, and 12 in the United States.

- b. Find the function that converts European dress sizes to U.S. dress sizes.
- c. Use part b. to find the dress sizes in the United States that correspond to 46, 52, 62, and 70.

Exercise:

Problem:

[T] The cost to remove a toxin from a lake is modeled by the function

$C(p) = 75p/(85 - p)$, where C is the cost (in thousands of dollars) and p is the amount of toxin in a small lake (measured in parts per billion [ppb]). This model is valid only when the amount of toxin is less than 85 ppb.

- a. Find the cost to remove 25 ppb, 40 ppb, and 50 ppb of the toxin from the lake.
- b. Find the inverse function. c. Use part b. to determine how much of the toxin is removed for \$50,000.

Solution:

- a. \$31,250, \$66,667, \$107,143 b. $(p = \frac{85C}{C+75})$ c. 34 ppb

Exercise:

Problem: [T] A race car is accelerating at a velocity given by

$v(t) = \frac{25}{4}t + 54$, where v is the velocity (in feet per second) at time t .

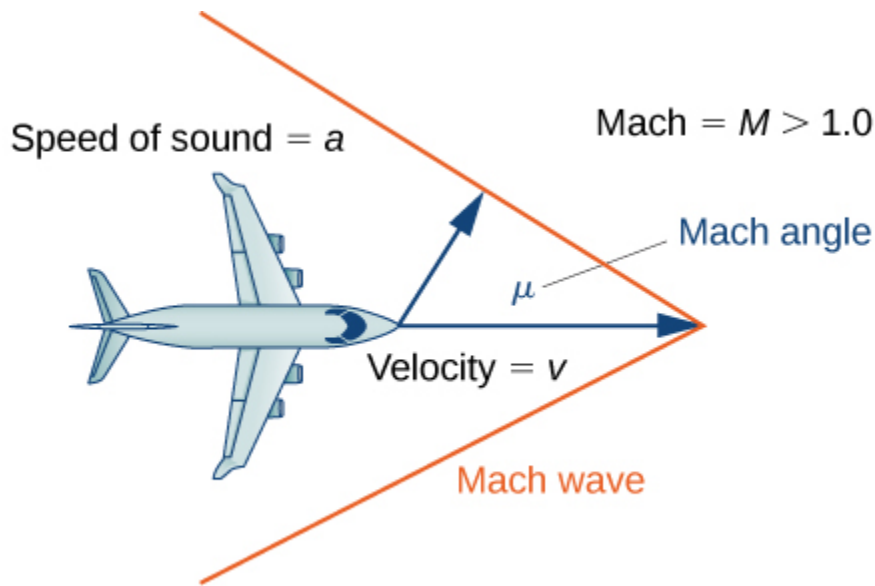
- a. Find the velocity of the car at 10 sec.
- b. Find the inverse function.
- c. Use part b. to determine how long it takes for the car to reach a speed of 150 ft/sec.

Exercise:

Problem:

[T] An airplane's Mach number M is the ratio of its speed to the speed of sound. When a plane is flying at a constant altitude, then its Mach angle is given by $\mu = 2\sin^{-1}\left(\frac{1}{M}\right)$.

Find the Mach angle (to the nearest degree) for the following Mach numbers.



- a. $M = 1.4$
- b. $M = 2.8$
- c. $M = 4.3$

Solution:

- a. $\sim 92^\circ$ b. $\sim 42^\circ$ c. $\sim 27^\circ$

Exercise:**Problem:**

[T] Using $\mu = 2\sin^{-1}\left(\frac{1}{M}\right)$, find the Mach number M for the following angles.

- a. $\mu = \frac{\pi}{6}$
- b. $\mu = \frac{2\pi}{7}$
- c. $\mu = \frac{3\pi}{8}$

Exercise:

Problem:

[T] The average temperature (in degrees Celsius) of a city in the northern United States can be modeled by the function

$T(x) = 5 + 18\sin\left[\frac{\pi}{6}(x - 4.6)\right]$, where x is time in months and $x = 1.00$ corresponds to January 1. Determine the month and day when the average temperature is 21°C .

Solution:

$x \approx 6.69, 8.51$; so, the temperature occurs on June 21 and August 15

Exercise:

Problem:

[T] The depth (in feet) of water at a dock changes with the rise and fall of tides. It is modeled by the function

$D(t) = 5\sin\left(\frac{\pi}{6}t - \frac{7\pi}{6}\right) + 8$, where t is the number of hours after midnight. Determine the first time after midnight when the depth is 11.75 ft.

Exercise:

Problem:

[T] An object moving in simple harmonic motion is modeled by the function

$s(t) = -6\cos\left(\frac{\pi t}{2}\right)$, where s is measured in inches and t is measured in seconds. Determine the first time when the distance moved is 4.5 in.

Solution:

~1.5 sec

Exercise:**Problem:**

[T] A local art gallery has a portrait 3 ft in height that is hung 2.5 ft above the eye level of an average person. The viewing angle θ can be modeled by the function

$\theta = \tan^{-1} \frac{5.5}{x} - \tan^{-1} \frac{2.5}{x}$, where x is the distance (in feet) from the portrait. Find the viewing angle when a person is 4 ft from the portrait.

Exercise:**Problem:**

[T] Use a calculator to evaluate $\tan^{-1}(\tan(2.1))$ and $\cos^{-1}(\cos(2.1))$. Explain the results of each.

Solution:

$\tan^{-1}(\tan(2.1)) \approx -1.0416$; the expression does not equal 2.1 since $2.1 > 1.57 = \frac{\pi}{2}$ —in other words, it is not in the restricted domain of $\tan x$. $\cos^{-1}(\cos(2.1)) = 2.1$, since 2.1 is in the restricted domain of $\cos x$.

Exercise:**Problem:**

[T] Use a calculator to evaluate $\sin(\sin^{-1}(-2))$ and $\tan(\tan^{-1}(-2))$. Explain the results of each.

Glossary

horizontal line test

a function f is one-to-one if and only if every horizontal line intersects the graph of f , at most, once

inverse function

for a function f , the inverse function f^{-1} satisfies $f^{-1}(y) = x$ if $f(x) = y$

inverse trigonometric functions

the inverses of the trigonometric functions are defined on restricted domains where they are one-to-one functions

one-to-one function

a function f is one-to-one if $f(x_1) \neq f(x_2)$ if $x_1 \neq x_2$

restricted domain

a subset of the domain of a function f

Exponential and Logarithmic Functions

- Identify the form of an exponential function.
- Explain the difference between the graphs of x^b and b^x .
- Recognize the significance of the number e .
- Identify the form of a logarithmic function.
- Explain the relationship between exponential and logarithmic functions.
- Describe how to calculate a logarithm to a different base.
- Identify the hyperbolic functions, their graphs, and basic identities.

In this section we examine exponential and logarithmic functions. We use the properties of these functions to solve equations involving exponential or logarithmic terms, and we study the meaning and importance of the number e . We also define hyperbolic and inverse hyperbolic functions, which involve combinations of exponential and logarithmic functions. (Note that we present alternative definitions of exponential and logarithmic functions in the chapter [Applications of Integrations](#), and prove that the functions have the same properties with either definition.)

Exponential Functions

Exponential functions arise in many applications. One common example is population growth.

For example, if a population starts with P_0 individuals and then grows at an annual rate of 2%, its population after 1 year is

Equation:

$$P(1) = P_0 + 0.02P_0 = P_0(1 + 0.02) = P_0(1.02).$$

Its population after 2 years is

Equation:

$$P(2) = P(1) + 0.02P(1) = P(1)(1.02) = P_0(1.02)^2.$$

In general, its population after t years is

Equation:

$$P(t) = P_0(1.02)^t,$$

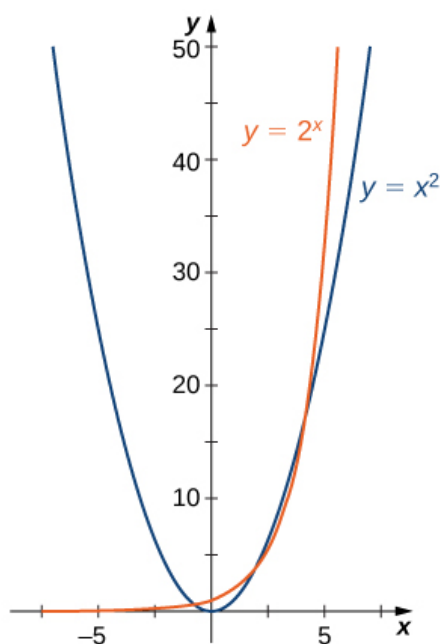
which is an exponential function. More generally, any function of the form $f(x) = b^x$, where $b > 0$, $b \neq 1$, is an exponential function with **base** b and **exponent** x . Exponential functions have constant bases and variable exponents. Note that a function of the form $f(x) = x^b$ for some constant b is not an exponential function but a power function.

To see the difference between an exponential function and a power function, we compare the functions $y = x^2$ and $y = 2^x$. In [\[link\]](#), we see that both 2^x and x^2 approach infinity as $x \rightarrow \infty$. Eventually, however, 2^x becomes larger than x^2 and grows more rapidly as $x \rightarrow \infty$. In the opposite direction, as $x \rightarrow -\infty$, $x^2 \rightarrow \infty$, whereas $2^x \rightarrow 0$. The line $y = 0$ is a horizontal asymptote for $y = 2^x$.

x	-3	-2	-1	0	1	2	3	4	5	6
x^2	9	4	1	0	1	4	9	16	25	36
2^x	1/8	1/4	1/2	1	2	4	8	16	32	64

Values of x^2 and 2^x

In [\[link\]](#), we graph both $y = x^2$ and $y = 2^x$ to show how the graphs differ.



Both 2^x and x^2 approach infinity as $x \rightarrow \infty$, but 2^x grows more rapidly than x^2 . As $x \rightarrow -\infty$, $x^2 \rightarrow \infty$, whereas $2^x \rightarrow 0$.

Evaluating Exponential Functions

Recall the properties of exponents: If x is a positive integer, then we define $b^x = b \cdot b \cdots b$ (with x factors of b). If x is a negative integer, then $x = -y$ for some positive integer y , and we define $b^x = b^{-y} = 1/b^y$. Also, b^0 is defined to be 1. If x is a rational number, then $x = p/q$, where p and q are integers and $b^x = b^{p/q} = \sqrt[q]{b^p}$. For example, $9^{3/2} = \sqrt{9^3} = 27$. However, how is b^x defined if x is an irrational number? For example, what do we mean by $2^{\sqrt{2}}$? This is too complex a question for

us to answer fully right now; however, we can make an approximation. In [\[link\]](#), we list some rational numbers approaching $\sqrt{2}$, and the values of 2^x for each rational number x are presented as well. We claim that if we choose rational numbers x getting closer and closer to $\sqrt{2}$, the values of 2^x get closer and closer to some number L . We define that number L to be $2^{\sqrt{2}}$.

x	1.4	1.41	1.414	1.4142	1.41421	1.414213
2^x	2.639	2.65737	2.66475	2.665119	2.665138	2.665143

Values of 2^x for a List of Rational Numbers Approximating $\sqrt{2}$

Example:

Exercise:

Problem:

Bacterial Growth

Suppose a particular population of bacteria is known to double in size every 4 hours. If a culture starts with 1000 bacteria, the number of bacteria after 4 hours is $n(4) = 1000 \cdot 2$. The number of bacteria after 8 hours is $n(8) = n(4) \cdot 2 = 1000 \cdot 2^2$. In general, the number of bacteria after $4m$ hours is $n(4m) = 1000 \cdot 2^m$. Letting $t = 4m$, we see that the number of bacteria after t hours is $n(t) = 1000 \cdot 2^{t/4}$. Find the number of bacteria after 6 hours, 10 hours, and 24 hours.

Solution:

The number of bacteria after 6 hours is given by $n(6) = 1000 \cdot 2^{6/4} \approx 2828$ bacteria. The number of bacteria after 10 hours is given by $n(10) = 1000 \cdot 2^{10/4} \approx 5657$ bacteria. The number of bacteria after 24 hours is given by $n(24) = 1000 \cdot 2^6 = 64,000$ bacteria.

Note:

Exercise:

Problem: Given the exponential function $f(x) = 100 \cdot 3^{x/2}$, evaluate $f(4)$ and $f(10)$.

Solution:

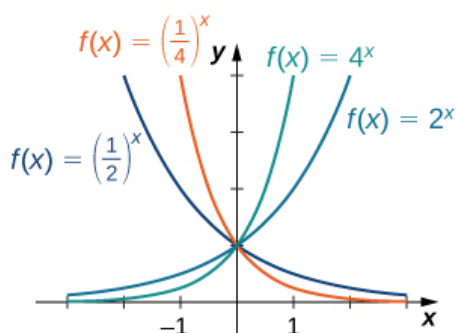
$f(4) = 900$; $f(10) = 24,300$.

Note:

Go to [World Population Balance](#) for another example of exponential population growth.

Graphing Exponential Functions

For any base $b > 0, b \neq 1$, the exponential function $f(x) = b^x$ is defined for all real numbers x and $b^x > 0$. Therefore, the domain of $f(x) = b^x$ is $(-\infty, \infty)$ and the range is $(0, \infty)$. To graph b^x , we note that for $b > 1$, b^x is increasing on $(-\infty, \infty)$ and $b^x \rightarrow \infty$ as $x \rightarrow \infty$, whereas $b^x \rightarrow 0$ as $x \rightarrow -\infty$. On the other hand, if $0 < b < 1$, $f(x) = b^x$ is decreasing on $(-\infty, \infty)$ and $b^x \rightarrow 0$ as $x \rightarrow \infty$ whereas $b^x \rightarrow \infty$ as $x \rightarrow -\infty$ ([link](#)).



If $b > 1$, then b^x is increasing on $(-\infty, \infty)$. If $0 < b < 1$, then b^x is decreasing on $(-\infty, \infty)$.

Note:

Visit this [site](#) for more exploration of the graphs of exponential functions.

Note that exponential functions satisfy the general laws of exponents. To remind you of these laws, we state them as rules.

Note:**Rule: Laws of Exponents**

For any constants $a > 0, b > 0$, and for all x and y ,

1. $b^x \cdot b^y = b^{x+y}$
2. $\frac{b^x}{b^y} = b^{x-y}$

3. $(b^x)^y = b^{xy}$
4. $(ab)^x = a^x b^x$
5. $\frac{a^x}{b^x} = \left(\frac{a}{b}\right)^x$

Example:

Exercise:

Problem:

Using the Laws of Exponents

Use the laws of exponents to simplify each of the following expressions.

- a. $\frac{(2x^{2/3})^3}{(4x^{-1/3})^2}$
- b. $\frac{(x^3y^{-1})^2}{(xy^2)^{-2}}$

Solution:

- a. We can simplify as follows:

Equation:

$$\frac{(2x^{2/3})^3}{(4x^{-1/3})^2} = \frac{2^3(x^{2/3})^3}{4^2(x^{-1/3})^2} = \frac{8x^2}{16x^{-2/3}} = \frac{x^2x^{2/3}}{2} = \frac{x^{8/3}}{2}.$$

- b. We can simplify as follows:

Equation:

$$\frac{(x^3y^{-1})^2}{(xy^2)^{-2}} = \frac{(x^3)^2(y^{-1})^2}{x^{-2}(y^2)^{-2}} = \frac{x^6y^{-2}}{x^{-2}y^{-4}} = x^6x^2y^{-2}y^4 = x^8y^2.$$

Note:

Exercise:

Problem: Use the laws of exponents to simplify $(6x^{-3}y^2) / (12x^{-4}y^5)$.

Solution:

$$x / (2y^3)$$

Hint

$$x^a/x^b = x^{a-b}$$

The Number e

A special type of exponential function appears frequently in real-world applications. To describe it, consider the following example of exponential growth, which arises from compounding interest in a savings account. Suppose a person invests P dollars in a savings account with an annual interest rate r , compounded annually. The amount of money after 1 year is

Equation:

$$A(1) = P + rP = P(1 + r).$$

The amount of money after 2 years is

Equation:

$$A(2) = A(1) + rA(1) = P(1 + r) + rP(1 + r) = P(1 + r)^2.$$

More generally, the amount after t years is

Equation:

$$A(t) = P(1 + r)^t.$$

If the money is compounded 2 times per year, the amount of money after half a year is

Equation:

$$A\left(\frac{1}{2}\right) = P + \left(\frac{r}{2}\right)P = P\left(1 + \left(\frac{r}{2}\right)\right).$$

The amount of money after 1 year is

Equation:

$$A(1) = A\left(\frac{1}{2}\right) + \left(\frac{r}{2}\right)A\left(\frac{1}{2}\right) = P\left(1 + \frac{r}{2}\right) + \frac{r}{2}\left(P\left(1 + \frac{r}{2}\right)\right) = P\left(1 + \frac{r}{2}\right)^2.$$

After t years, the amount of money in the account is

Equation:

$$A(t) = P\left(1 + \frac{r}{2}\right)^{2t}.$$

More generally, if the money is compounded n times per year, the amount of money in the account after t years is given by the function

Equation:

$$A(t) = P\left(1 + \frac{r}{n}\right)^{nt}.$$

What happens as $n \rightarrow \infty$? To answer this question, we let $m = n/r$ and write

Equation:

$$\left(1 + \frac{r}{n}\right)^{nt} = \left(1 + \frac{1}{m}\right)^{mrt},$$

and examine the behavior of $\left(1 + 1/m\right)^m$ as $m \rightarrow \infty$, using a table of values ([link](#)).

m	10	100	1000	10,000	100,000	1,000,000
$\left(1 + \frac{1}{m}\right)^m$	2.5937	2.7048	2.71692	2.71815	2.718268	2.718280

Values of $\left(1 + \frac{1}{m}\right)^m$ as $m \rightarrow \infty$

Looking at this table, it appears that $\left(1 + 1/m\right)^m$ is approaching a number between 2.7 and 2.8 as $m \rightarrow \infty$. In fact, $\left(1 + 1/m\right)^m$ does approach some number as $m \rightarrow \infty$. We call this **number** e . To six decimal places of accuracy,

Equation:

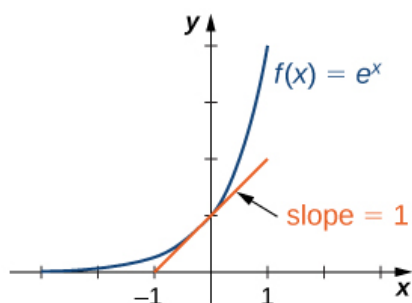
$$e \approx 2.718282.$$

The letter e was first used to represent this number by the Swiss mathematician Leonhard Euler during the 1720s. Although Euler did not discover the number, he showed many important connections between e and logarithmic functions. We still use the notation e today to honor Euler's work because it appears in many areas of mathematics and because we can use it in many practical applications.

Returning to our savings account example, we can conclude that if a person puts P dollars in an account at an annual interest rate r , compounded continuously, then $A(t) = Pe^{rt}$. This function may be familiar. Since functions involving base e arise often in applications, we call the function $f(x) = e^x$ the **natural exponential function**. Not only is this function interesting because of the definition of the number e , but also, as discussed next, its graph has an important property.

Since $e > 1$, we know e^x is increasing on $(-\infty, \infty)$. In [link](#), we show a graph of $f(x) = e^x$ along with a *tangent line* to the graph of at $x = 0$. We give a precise definition of tangent line in the next chapter; but, informally, we say a tangent line to a graph of f at $x = a$ is a line that passes through the point $(a, f(a))$ and has the same "slope" as f at that point. The function $f(x) = e^x$ is the only exponential function b^x with tangent line at $x = 0$ that has a slope of 1. As we see later in the text,

having this property makes the natural exponential function the most simple exponential function to use in many instances.



The graph of $f(x) = e^x$ has a tangent line with slope 1 at $x = 0$.

Example:

Exercise:

Problem:

Compounding Interest

Suppose \$500 is invested in an account at an annual interest rate of $r = 5.5\%$, compounded continuously.

- Let t denote the number of years after the initial investment and $A(t)$ denote the amount of money in the account at time t . Find a formula for $A(t)$.
- Find the amount of money in the account after 10 years and after 20 years.

Solution:

- If P dollars are invested in an account at an annual interest rate r , compounded continuously, then $A(t) = Pe^{rt}$. Here $P = \$500$ and $r = 0.055$. Therefore, $A(t) = 500e^{0.055t}$.
- After 10 years, the amount of money in the account is

Equation:

$$A(10) = 500e^{0.055 \cdot 10} = 500e^{0.55} \approx \$866.63.$$

After 20 years, the amount of money in the account is

Equation:

$$A(20) = 500e^{0.055 \cdot 20} = 500e^{1.1} \approx \$1,502.08.$$

Note:

Exercise:

Problem:

If \$750 is invested in an account at an annual interest rate of 4%, compounded continuously, find a formula for the amount of money in the account after t years. Find the amount of money after 30 years.

Solution:

$A(t) = 750e^{0.04t}$. After 30 years, there will be approximately \$2,490.09.

Hint

$$A(t) = Pe^{rt}.$$

Logarithmic Functions

Using our understanding of exponential functions, we can discuss their inverses, which are the logarithmic functions. These come in handy when we need to consider any phenomenon that varies over a wide range of values, such as pH in chemistry or decibels in sound levels.

The exponential function $f(x) = b^x$ is one-to-one, with domain $(-\infty, \infty)$ and range $(0, \infty)$. Therefore, it has an inverse function, called the *logarithmic function with base b* . For any $b > 0, b \neq 1$, the logarithmic function with base b , denoted $\log_b$, has domain $(0, \infty)$ and range $(-\infty, \infty)$, and satisfies

Equation:

$$\log_b(x) = y \text{ if and only if } b^y = x.$$

For example,

Equation:

$$\begin{array}{lll} \log_2(8) = 3 & \text{since} & 2^3 = 8, \\ \log_{10}\left(\frac{1}{100}\right) = -2 & \text{since} & 10^{-2} = \frac{1}{10^2} = \frac{1}{100}, \\ \log_b(1) = 0 & \text{since} & b^0 = 1 \text{ for any base } b > 0. \end{array}$$

Furthermore, since $y = \log_b(x)$ and $y = b^x$ are inverse functions,

Equation:

$$\log_b(b^x) = x \text{ and } b^{\log_b(x)} = x.$$

The most commonly used logarithmic function is the function $\log_e$. Since this function uses natural e as its base, it is called the **natural logarithm**. Here we use the notation $\ln(x)$ or $\ln x$ to mean $\log_e(x)$. For example,

Equation:

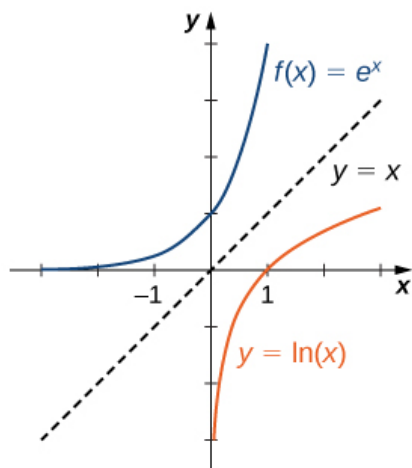
$$\ln(e) = \log_e(e) = 1, \ln(e^3) = \log_e(e^3) = 3, \ln(1) = \log_e(1) = 0.$$

Since the functions $f(x) = e^x$ and $g(x) = \ln(x)$ are inverses of each other,

Equation:

$$\ln(e^x) = x \text{ and } e^{\ln x} = x,$$

and their graphs are symmetric about the line $y = x$ ([link](#)).

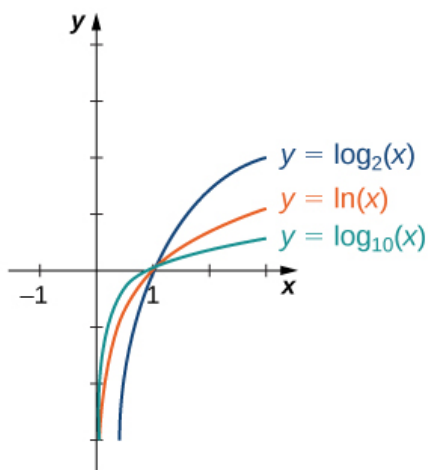


The functions $y = e^x$ and $y = \ln(x)$ are inverses of each other, so their graphs are symmetric about the line $y = x$.

Note:

At this [site](#) you can see an example of a base-10 logarithmic scale.

In general, for any base $b > 0, b \neq 1$, the function $g(x) = \log_b(x)$ is symmetric about the line $y = x$ with the function $f(x) = b^x$. Using this fact and the graphs of the exponential functions, we graph functions $\log_b$ for several values of $b > 1$ ([link](#)).



Graphs of $y = \log_b(x)$ are depicted for $b = 2, e, 10$.

Before solving some equations involving exponential and logarithmic functions, let's review the basic properties of logarithms.

Note:

Rule: Properties of Logarithms

If $a, b, c > 0$, $b \neq 1$, and r is any real number, then

Equation:

1. $\log_b(ac) = \log_b(a) + \log_b(c)$ (Product property)
2. $\log_b\left(\frac{a}{c}\right) = \log_b(a) - \log_b(c)$ (Quotient property)
3. $\log_b(a^r) = r\log_b(a)$ (Power property)

Example:

Exercise:

Problem:

Solving Equations Involving Exponential Functions

Solve each of the following equations for x .

- a. $5^x = 2$
- b. $e^x + 6e^{-x} = 5$

Solution:

a. Applying the natural logarithm function to both sides of the equation, we have

Equation:

$$\ln 5^x = \ln 2.$$

Using the power property of logarithms,

Equation:

$$x \ln 5 = \ln 2.$$

Therefore, $x = \ln 2 / \ln 5$.

b. Multiplying both sides of the equation by e^x , we arrive at the equation

Equation:

$$e^{2x} + 6 = 5e^x.$$

Rewriting this equation as

Equation:

$$e^{2x} - 5e^x + 6 = 0,$$

we can then rewrite it as a quadratic equation in e^x :

Equation:

$$(e^x)^2 - 5(e^x) + 6 = 0.$$

Now we can solve the quadratic equation. Factoring this equation, we obtain

Equation:

$$(e^x - 3)(e^x - 2) = 0.$$

Therefore, the solutions satisfy $e^x = 3$ and $e^x = 2$. Taking the natural logarithm of both sides gives us the solutions $x = \ln 3, \ln 2$.

Note:

Exercise:

Problem: Solve $e^{2x}/(3 + e^{2x}) = 1/2$.

Solution:

$$x = \frac{\ln 3}{2}$$

Hint

First solve the equation for e^{2x} .

Example:**Exercise:****Problem:****Solving Equations Involving Logarithmic Functions**

Solve each of the following equations for x .

- a. $\ln\left(\frac{1}{x}\right) = 4$
- b. $\log_{10}\sqrt{x} + \log_{10}x = 2$
- c. $\ln(2x) - 3\ln(x^2) = 0$

Solution:

- a. By the definition of the natural logarithm function,

Equation:

$$\ln\left(\frac{1}{x}\right) = 4 \text{ if and only if } e^4 = \frac{1}{x}.$$

Therefore, the solution is $x = 1/e^4$.

- b. Using the product and power properties of logarithmic functions, rewrite the left-hand side of the equation as

Equation:

$$\log_{10}\sqrt{x} + \log_{10}x = \log_{10}x\sqrt{x} = \log_{10}x^{3/2} = \frac{3}{2}\log_{10}x.$$

Therefore, the equation can be rewritten as

Equation:

$$\frac{3}{2}\log_{10}x = 2 \text{ or } \log_{10}x = \frac{4}{3}.$$

The solution is $x = 10^{4/3} = 10\sqrt[3]{10}$.

- c. Using the power property of logarithmic functions, we can rewrite the equation as $\ln(2x) - \ln(x^6) = 0$.

Using the quotient property, this becomes

Equation:

$$\ln \left(\frac{2}{x^5} \right) = 0.$$

Therefore, $2/x^5 = 1$, which implies $x = \sqrt[5]{2}$. We should then check for any extraneous solutions.

Note:

Exercise:

Problem: Solve $\ln(x^3) - 4\ln(x) = 1$.

Solution:

$$x = \frac{1}{e}$$

Hint

First use the power property, then use the product property of logarithms.

When evaluating a logarithmic function with a calculator, you may have noticed that the only options are $\log_{10}$ or $\log$, called the *common logarithm*, or $\ln$, which is the natural logarithm. However, exponential functions and logarithm functions can be expressed in terms of any desired base b . If you need to use a calculator to evaluate an expression with a different base, you can apply the change-of-base formulas first. Using this change of base, we typically write a given exponential or logarithmic function in terms of the natural exponential and natural logarithmic functions.

Note:

Rule: Change-of-Base Formulas

Let $a > 0$, $b > 0$, and $a \neq 1$, $b \neq 1$.

1. $a^x = b^{x \log_b a}$ for any real number x .
If $b = e$, this equation reduces to $a^x = e^{x \log_e a} = e^{x \ln a}$.
2. $\log_a x = \frac{\log_b x}{\log_b a}$ for any real number $x > 0$.
If $b = e$, this equation reduces to $\log_a x = \frac{\ln x}{\ln a}$.

Proof

For the first change-of-base formula, we begin by making use of the power property of logarithmic functions. We know that for any base $b > 0, b \neq 1, \log_b(a^x) = x \log_b a$. Therefore,

Equation:

$$b^{\log_b(a^x)} = b^{x \log_b a}.$$

In addition, we know that b^x and $\log_b(x)$ are inverse functions. Therefore,

Equation:

$$b^{\log_b(a^x)} = a^x.$$

Combining these last two equalities, we conclude that $a^x = b^{x \log_b a}$.

To prove the second property, we show that

Equation:

$$(\log_b a) \cdot (\log_a x) = \log_b x.$$

Let $u = \log_b a, v = \log_a x$, and $w = \log_b x$. We will show that $u \cdot v = w$. By the definition of logarithmic functions, we know that $b^u = a, a^v = x$, and $b^w = x$. From the previous equations, we see that

Equation:

$$b^{uv} = (b^u)^v = a^v = x = b^w.$$

Therefore, $b^{uv} = b^w$. Since exponential functions are one-to-one, we can conclude that $u \cdot v = w$.

□

Example:

Exercise:

Problem:

Changing Bases

Use a calculating utility to evaluate $\log_3 7$ with the change-of-base formula presented earlier.

Solution:

Use the second equation with $a = 3$ and $e = 3$:

$$\log_3 7 = \frac{\ln 7}{\ln 3} \approx 1.77124.$$

Note:

Exercise:

Problem: Use the change-of-base formula and a calculating utility to evaluate $\log_4 6$.

Solution:

1.29248

Hint

Use the change of base to rewrite this expression in terms of expressions involving the natural logarithm function.

Example:**Exercise:**

Problem:

Chapter Opener: The Richter Scale for Earthquakes

(credit: modification of work by
Robb Hannawacker, NPS)

In 1935, Charles Richter developed a scale (now known as the *Richter scale*) to measure the magnitude of an earthquake. The scale is a base-10 logarithmic scale, and it can be described as follows: Consider one earthquake with magnitude R_1 on the Richter scale and a second earthquake with magnitude R_2 on the Richter scale. Suppose $R_1 > R_2$, which means the earthquake of magnitude R_1 is stronger, but how much stronger is it than the other earthquake? A way of measuring the intensity of an earthquake is by using a seismograph to measure the amplitude of the earthquake waves. If A_1 is the amplitude measured for the first earthquake and A_2 is the amplitude measured for the second earthquake, then the amplitudes and magnitudes of the two earthquakes satisfy the following equation:

Equation:

$$R_1 - R_2 = \log_{10} \left(\frac{A_1}{A_2} \right).$$

Consider an earthquake that measures 8 on the Richter scale and an earthquake that measures 7 on the Richter scale. Then,

Equation:

$$8 - 7 = \log_{10} \left(\frac{A_1}{A_2} \right).$$

Therefore,

Equation:

$$\log_{10} \left(\frac{A_1}{A_2} \right) = 1,$$

which implies $A_1/A_2 = 10$ or $A_1 = 10A_2$. Since A_1 is 10 times the size of A_2 , we say that the first earthquake is 10 times as intense as the second earthquake. On the other hand, if one earthquake measures 8 on the Richter scale and another measures 6, then the relative intensity of the two earthquakes satisfies the equation

Equation:

$$\log_{10} \left(\frac{A_1}{A_2} \right) = 8 - 6 = 2.$$

Therefore, $A_1 = 100A_2$. That is, the first earthquake is 100 times more intense than the second earthquake.

How can we use logarithmic functions to compare the relative severity of the magnitude 9 earthquake in Japan in 2011 with the magnitude 7.3 earthquake in Haiti in 2010?

Solution:

To compare the Japan and Haiti earthquakes, we can use an equation presented earlier:

$$9 - 7.3 = \log_{10} \left(\frac{A_1}{A_2} \right).$$

Therefore, $A_1/A_2 = 10^{1.7}$, and we conclude that the earthquake in Japan was approximately 50 times more intense than the earthquake in Haiti.

Note:

Exercise:

Problem:

Compare the relative severity of a magnitude 8.4 earthquake with a magnitude 7.4 earthquake.

Solution:

The magnitude 8.4 earthquake is roughly 10 times as severe as the magnitude 7.4 earthquake.

Hint

$$R_1 - R_2 = \log_{10}(A_1/A_2).$$

Hyperbolic Functions

The hyperbolic functions are defined in terms of certain combinations of e^x and e^{-x} . These functions arise naturally in various engineering and physics applications, including the study of water waves and vibrations of elastic membranes. Another common use for a hyperbolic function is the representation of a hanging chain or cable, also known as a catenary ([link](#)). If we introduce a coordinate system so that the low point of the chain lies along the y -axis, we can describe the height of the chain in terms of a hyperbolic function. First, we define the **hyperbolic functions**.



The shape of a strand of silk in a spider's web can be described in terms of a hyperbolic function. The same shape applies to a chain or cable hanging from two supports with only its own weight. (credit: "Mtpaley", Wikimedia Commons)

Note:

Definition

Hyperbolic cosine

Equation:

$$\cosh x = \frac{e^x + e^{-x}}{2}$$

Hyperbolic sine**Equation:**

$$\sinh x = \frac{e^x - e^{-x}}{2}$$

Hyperbolic tangent**Equation:**

$$\tanh x = \frac{\sinh x}{\cosh x} = \frac{e^x - e^{-x}}{e^x + e^{-x}}$$

Hyperbolic cosecant**Equation:**

$$\operatorname{csch} x = \frac{1}{\sinh x} = \frac{2}{e^x - e^{-x}}$$

Hyperbolic secant**Equation:**

$$\operatorname{sech} x = \frac{1}{\cosh x} = \frac{2}{e^x + e^{-x}}$$

Hyperbolic cotangent**Equation:**

$$\operatorname{coth} x = \frac{\cosh x}{\sinh x} = \frac{e^x + e^{-x}}{e^x - e^{-x}}$$

The name *cosh* rhymes with “gosh,” whereas the name *sinh* is pronounced “cinch.” *Tanh*, *sech*, *csch*, and *coth* are pronounced “tanch,” “seech,” “coseech,” and “cotanch,” respectively.

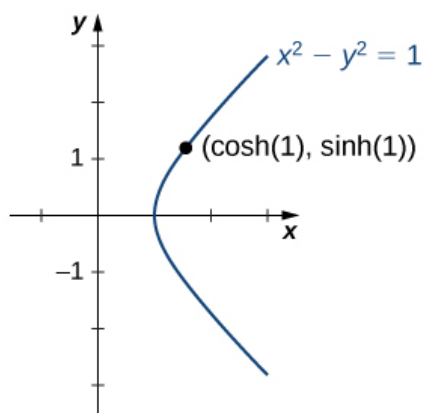
Using the definition of $\cosh(x)$ and principles of physics, it can be shown that the height of a hanging chain, such as the one in [\[link\]](#), can be described by the function $h(x) = a \cosh(x/a) + c$ for certain constants a and c .

But why are these functions called *hyperbolic functions*? To answer this question, consider the quantity $\cosh^2 t - \sinh^2 t$. Using the definition of $\cosh$ and $\sinh$, we see that

Equation:

$$\cosh^2 t - \sinh^2 t = \frac{e^{2t} + 2 + e^{-2t}}{4} - \frac{e^{2t} - 2 + e^{-2t}}{4} = 1.$$

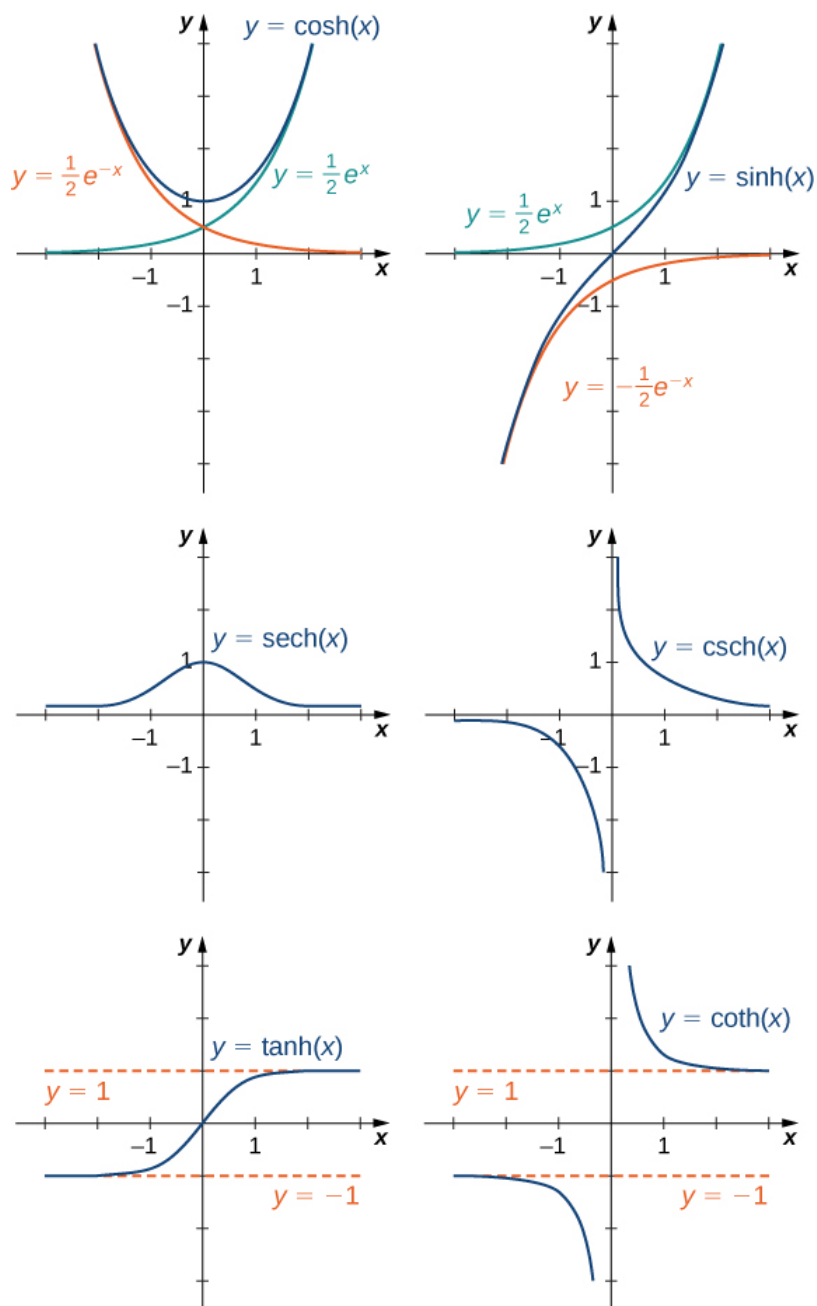
This identity is the analog of the trigonometric identity $\cos^2 t + \sin^2 t = 1$. Here, given a value t , the point $(x, y) = (\cosh t, \sinh t)$ lies on the unit hyperbola $x^2 - y^2 = 1$ ([\[link\]](#)).



The unit hyperbola
 $\cosh^2 t - \sinh^2 t = 1$.

Graphs of Hyperbolic Functions

To graph $\cosh x$ and $\sinh x$, we make use of the fact that both functions approach $(1/2)e^x$ as $x \rightarrow \infty$, since $e^{-x} \rightarrow 0$ as $x \rightarrow \infty$. As $x \rightarrow -\infty$, $\cosh x$ approaches $1/2e^{-x}$, whereas $\sinh x$ approaches $-1/2e^{-x}$. Therefore, using the graphs of $1/2e^x$, $1/2e^{-x}$, and $-1/2e^{-x}$ as guides, we graph $\cosh x$ and $\sinh x$. To graph $\tanh x$, we use the fact that $\tanh(0) = 0$, $-1 < \tanh(x) < 1$ for all x , $\tanh x \rightarrow 1$ as $x \rightarrow \infty$, and $\tanh x \rightarrow -1$ as $x \rightarrow -\infty$. The graphs of the other three hyperbolic functions can be sketched using the graphs of $\cosh x$, $\sinh x$, and $\tanh x$ ([link](#)).



The hyperbolic functions involve combinations of e^x and e^{-x} .

Identities Involving Hyperbolic Functions

The identity $\cosh^2 t - \sinh^2 t$, shown in [\[link\]](#), is one of several identities involving the hyperbolic functions, some of which are listed next. The first four properties follow easily from the definitions of

hyperbolic sine and hyperbolic cosine. Except for some differences in signs, most of these properties are analogous to identities for trigonometric functions.

Note:

Rule: Identities Involving Hyperbolic Functions

1. $\cosh(-x) = \cosh x$
2. $\sinh(-x) = -\sinh x$
3. $\cosh x + \sinh x = e^x$
4. $\cosh x - \sinh x = e^{-x}$
5. $\cosh^2 x - \sinh^2 x = 1$
6. $1 - \tanh^2 x = \operatorname{sech}^2 x$
7. $\coth^2 x - 1 = \operatorname{csch}^2 x$
8. $\sinh(x \pm y) = \sinh x \cosh y \pm \cosh x \sinh y$
9. $\cosh(x \pm y) = \cosh x \cosh y \pm \sinh x \sinh y$

Example:

Exercise:

Problem:

Evaluating Hyperbolic Functions

- a. Simplify $\sinh(5 \ln x)$.
- b. If $\sinh x = 3/4$, find the values of the remaining five hyperbolic functions.

Solution:

- a. Using the definition of the $\sinh$ function, we write

Equation:

$$\sinh(5 \ln x) = \frac{e^{5 \ln x} - e^{-5 \ln x}}{2} = \frac{e^{\ln(x^5)} - e^{\ln(x^{-5})}}{2} = \frac{x^5 - x^{-5}}{2}.$$

- b. Using the identity $\cosh^2 x - \sinh^2 x = 1$, we see that

Equation:

$$\cosh^2 x = 1 + \left(\frac{3}{4}\right)^2 = \frac{25}{16}.$$

Since $\cosh x \geq 1$ for all x , we must have $\cosh x = 5/4$. Then, using the definitions for the other hyperbolic functions, we conclude that $\tanh x = 3/5$, $\operatorname{csch} x = 4/3$, $\operatorname{sech} x = 4/5$, and $\coth x = 5/3$.

Note:

Exercise:

Problem: Simplify $\cosh(2\ln x)$.

Solution:

$$(x^2 + x^{-2})/2$$

Hint

Use the definition of the cosh function and the power property of logarithm functions.

Inverse Hyperbolic Functions

From the graphs of the hyperbolic functions, we see that all of them are one-to-one except $\cosh x$ and $\operatorname{sech} x$. If we restrict the domains of these two functions to the interval $[0, \infty)$, then all the hyperbolic functions are one-to-one, and we can define the **inverse hyperbolic functions**. Since the hyperbolic functions themselves involve exponential functions, the inverse hyperbolic functions involve logarithmic functions.

Note:

Definition

Inverse Hyperbolic Functions

Equation:

$$\begin{aligned}\sinh^{-1}x &= \operatorname{arcsinh} x = \ln(x + \sqrt{x^2 + 1}) & \cosh^{-1}x &= \operatorname{arccosh} x = \ln(x + \sqrt{x^2 - 1}) \\ \tanh^{-1}x &= \operatorname{arctanh} x = \frac{1}{2} \ln\left(\frac{1+x}{1-x}\right) & \coth^{-1}x &= \operatorname{arccot} x = \frac{1}{2} \ln\left(\frac{x+1}{x-1}\right) \\ \operatorname{sech}^{-1}x &= \operatorname{arcsech} x = \ln\left(\frac{1+\sqrt{1-x^2}}{x}\right) & \operatorname{csch}^{-1}x &= \operatorname{arccsch} x = \ln\left(\frac{1}{x} + \frac{\sqrt{1+x^2}}{|x|}\right)\end{aligned}$$

Let's look at how to derive the first equation. The others follow similarly. Suppose $y = \sinh^{-1}x$. Then, $x = \sinh y$ and, by the definition of the hyperbolic sine function, $x = \frac{e^y - e^{-y}}{2}$. Therefore,

Equation:

$$e^y - 2x - e^{-y} = 0.$$

Multiplying this equation by e^y , we obtain

Equation:

$$e^{2y} - 2xe^y - 1 = 0.$$

This can be solved like a quadratic equation, with the solution

Equation:

$$e^y = \frac{2x \pm \sqrt{4x^2 + 4}}{2} = x \pm \sqrt{x^2 + 1}.$$

Since $e^y > 0$, the only solution is the one with the positive sign. Applying the natural logarithm to both sides of the equation, we conclude that

Equation:

$$y = \ln \left(x + \sqrt{x^2 + 1} \right).$$

Example:

Exercise:

Problem:

Evaluating Inverse Hyperbolic Functions

Evaluate each of the following expressions.

Equation:

$$\sinh^{-1}(2)$$

Equation:

$$\tanh^{-1}(1/4)$$

Solution:

$$\sinh^{-1}(2) = \ln \left(2 + \sqrt{2^2 + 1} \right) = \ln \left(2 + \sqrt{5} \right) \approx 1.4436$$

$$\tanh^{-1}(1/4) = \frac{1}{2} \ln \left(\frac{1+1/4}{1-1/4} \right) = \frac{1}{2} \ln \left(\frac{5/4}{3/4} \right) = \frac{1}{2} \ln \left(\frac{5}{3} \right) \approx 0.2554$$

Note:

Exercise:

Problem: Evaluate $\tanh^{-1}(1/2)$.

Solution:

$$\frac{1}{2} \ln(3) \approx 0.5493.$$

Hint

Use the definition of $\tanh^{-1}x$ and simplify.

Key Concepts

- The exponential function $y = b^x$ is increasing if $b > 1$ and decreasing if $0 < b < 1$. Its domain is $(-\infty, \infty)$ and its range is $(0, \infty)$.
- The logarithmic function $y = \log_b(x)$ is the inverse of $y = b^x$. Its domain is $(0, \infty)$ and its range is $(-\infty, \infty)$.
- The natural exponential function is $y = e^x$ and the natural logarithmic function is $y = \ln x = \log_e x$.
- Given an exponential function or logarithmic function in base a , we can make a change of base to convert this function to any base $b > 0, b \neq 1$. We typically convert to base e .
- The hyperbolic functions involve combinations of the exponential functions e^x and e^{-x} . As a result, the inverse hyperbolic functions involve the natural logarithm.

For the following exercises, evaluate the given exponential functions as indicated, accurate to two significant digits after the decimal.

Exercise:

Problem: $f(x) = 5^x$ a. $x = 3$ b. $x = \frac{1}{2}$ c. $x = \sqrt{2}$

Solution:

a. 125 b. 2.24 c. 9.74

Exercise:

Problem: $f(x) = (0.3)^x$ a. $x = -1$ b. $x = 4$ c. $x = -1.5$

Exercise:

Problem: $f(x) = 10^x$ a. $x = -2$ b. $x = 4$ c. $x = \frac{5}{3}$

Solution:

a. 0.01 b. 10,000 c. 46.42

Exercise:

Problem: $f(x) = e^x$ a. $x = 2$ b. $x = -3.2$ c. $x = \pi$

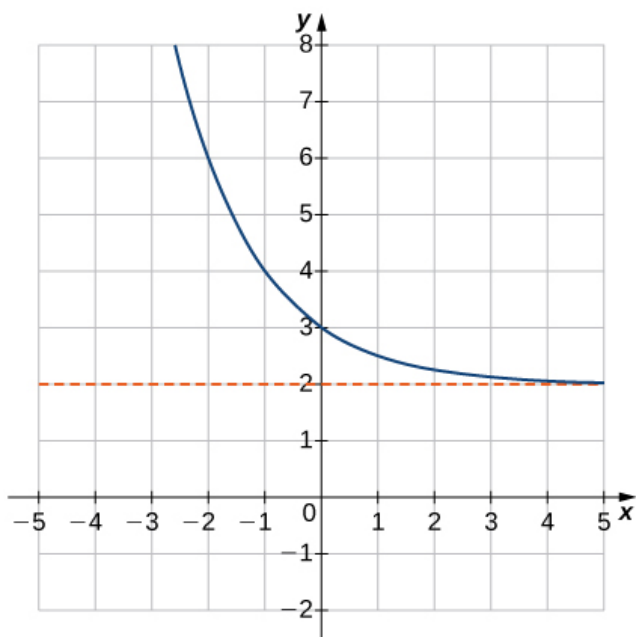
For the following exercises, match the exponential equation to the correct graph.

- a. $y = 4^{-x}$
- b. $y = 3^{x-1}$

- c. $y = 2^{x+1}$
- d. $y = \left(\frac{1}{2}\right)^x + 2$
- e. $y = -3^{-x}$
- f. $y = 1 - 5^x$

Exercise:

Problem:

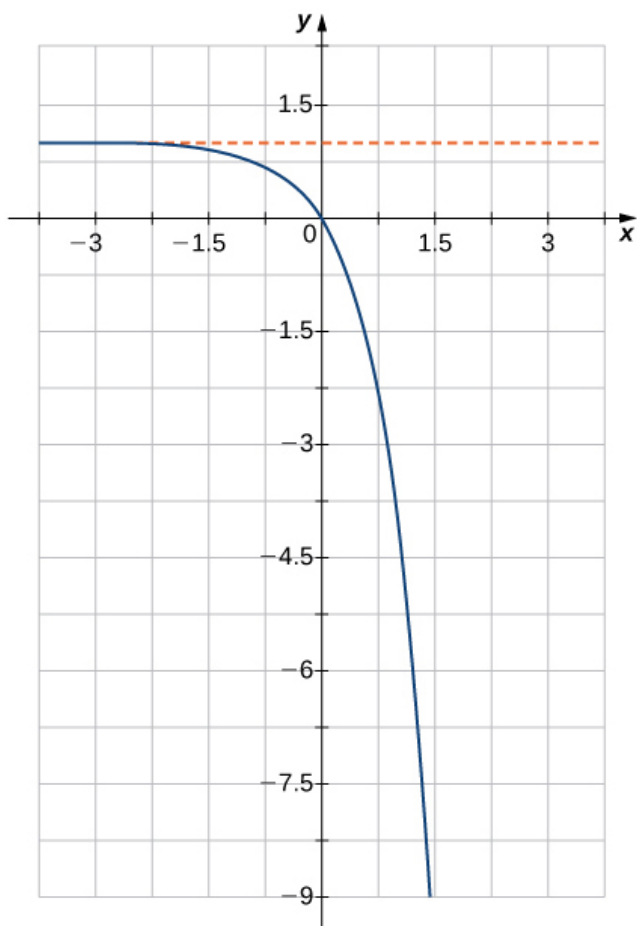


Solution:

d

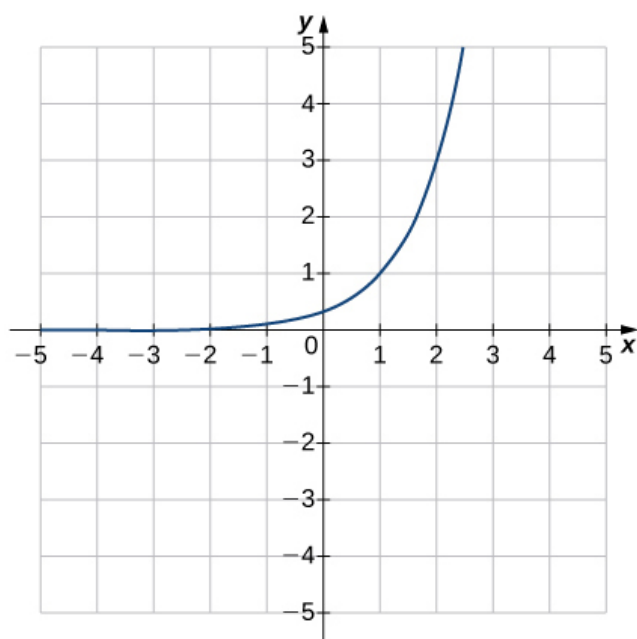
Exercise:

Problem:



Exercise:

Problem:

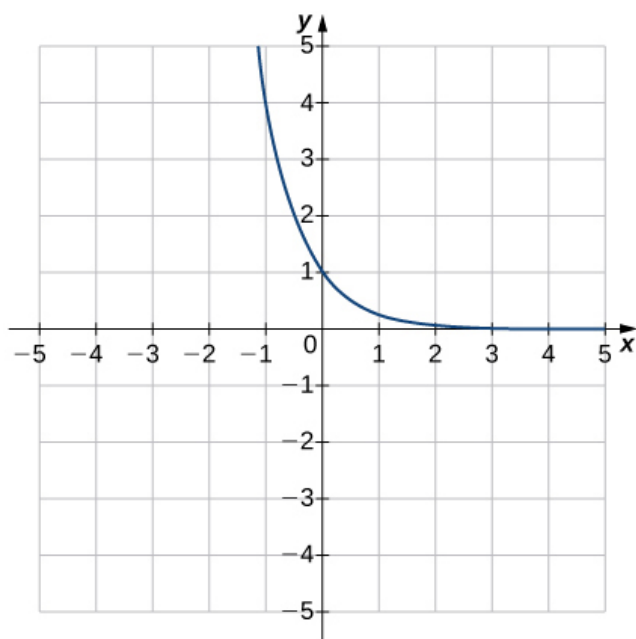


Solution:

b

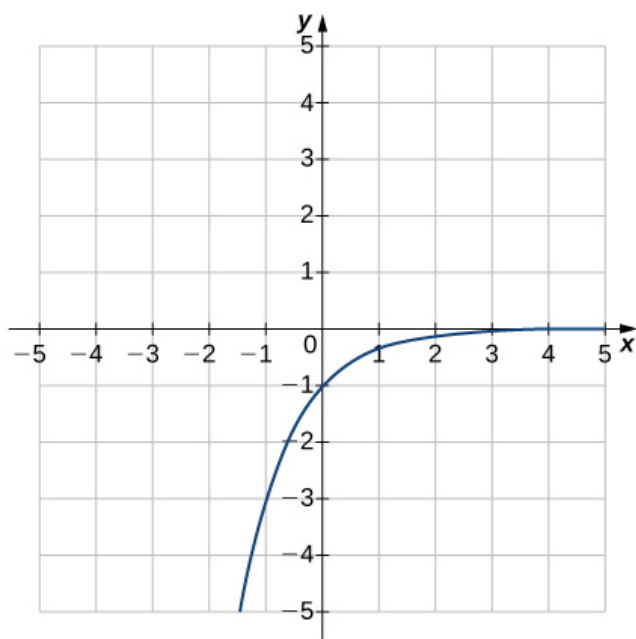
Exercise:

Problem:



Exercise:

Problem:

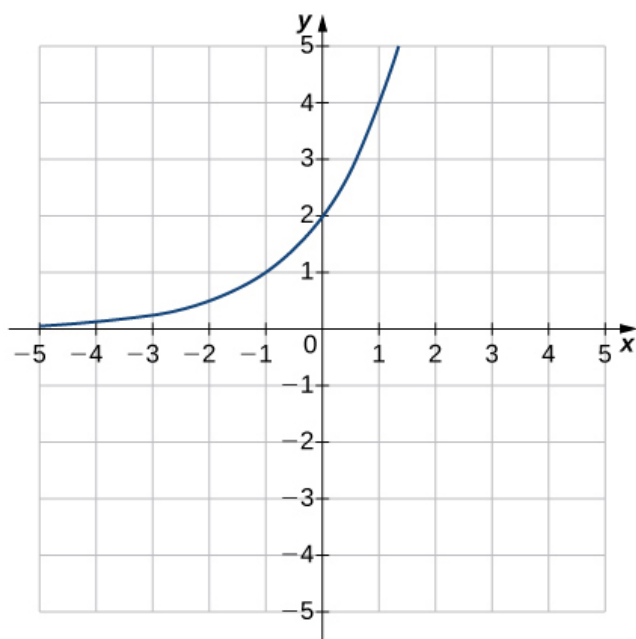


Solution:

e

Exercise:

Problem:



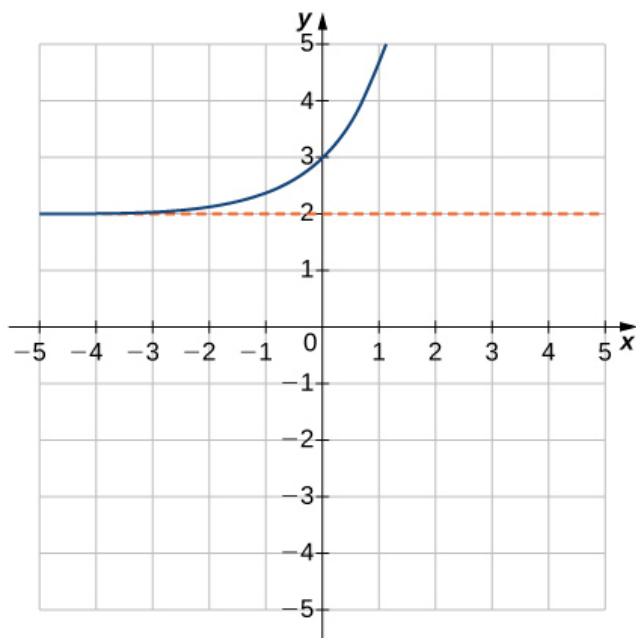
For the following exercises, sketch the graph of the exponential function. Determine the domain, range, and horizontal asymptote.

Exercise:

Problem: $f(x) = e^x + 2$

Solution:

Domain: all real numbers, range: $(2, \infty)$, $y = 2$



Exercise:

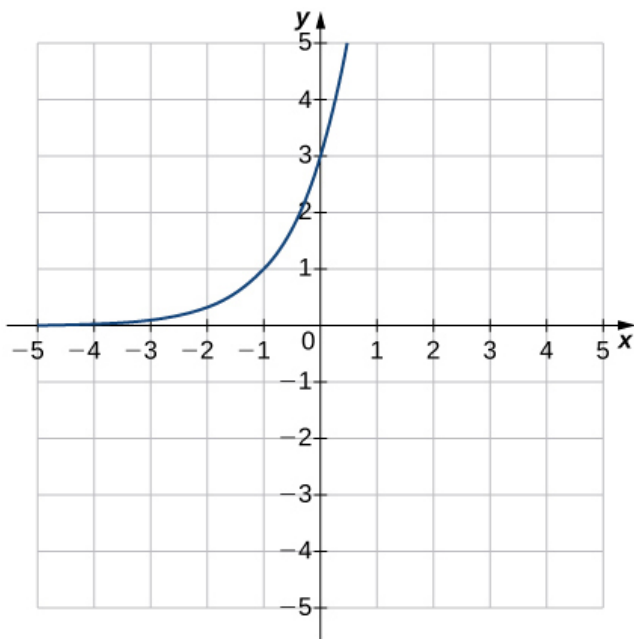
Problem: $f(x) = -2^x$

Exercise:

Problem: $f(x) = 3^{x+1}$

Solution:

Domain: all real numbers, range: $(0, \infty)$, $y = 0$



Exercise:

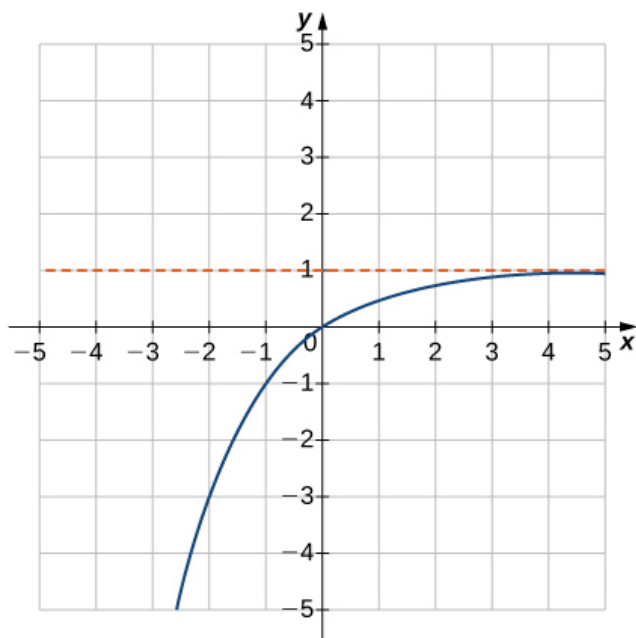
Problem: $f(x) = 4^x - 1$

Exercise:

Problem: $f(x) = 1 - 2^{-x}$

Solution:

Domain: all real numbers, range: $(-\infty, 1)$, $y = 1$



Exercise:

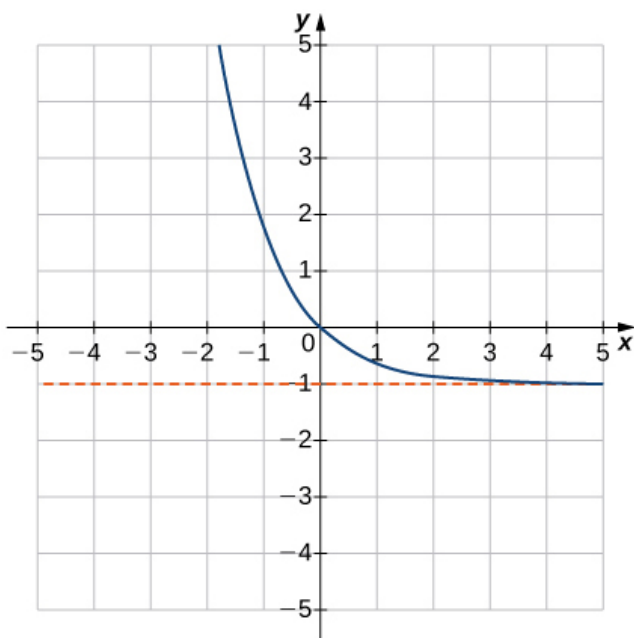
Problem: $f(x) = 5^{x+1} + 2$

Exercise:

Problem: $f(x) = e^{-x} - 1$

Solution:

Domain: all real numbers, range: $(-1, \infty)$, $y = -1$



For the following exercises, write the equation in equivalent exponential form.

Exercise:

Problem: $\log_3 81 = 4$

Exercise:

Problem: $\log_8 2 = \frac{1}{3}$

Solution:

$$8^{1/3} = 2$$

Exercise:

Problem: $\log_5 1 = 0$

Exercise:

Problem: $\log_5 25 = 2$

Solution:

$$5^2 = 25$$

Exercise:

Problem: $\log 0.1 = -1$

Exercise:

Problem: $\ln\left(\frac{1}{e^3}\right) = -3$

Solution:

$$e^{-3} = \frac{1}{e^3}$$

Exercise:

Problem: $\log_9 3 = 0.5$

Exercise:

Problem: $\ln 1 = 0$

Solution:

$$e^0 = 1$$

For the following exercises, write the equation in equivalent logarithmic form.

Exercise:

Problem: $2^3 = 8$

Exercise:

Problem: $4^{-2} = \frac{1}{16}$

Solution:

$$\log_4\left(\frac{1}{16}\right) = -2$$

Exercise:

Problem: $10^2 = 100$

Exercise:

Problem: $9^0 = 1$

Solution:

$$\log_9 1 = 0$$

Exercise:

Problem: $\left(\frac{1}{3}\right)^3 = \frac{1}{27}$

Exercise:

Problem: $\sqrt[3]{64} = 4$

Solution:

$$\log_{64} 4 = \frac{1}{3}$$

Exercise:

Problem: $e^x = y$

Exercise:

Problem: $9^y = 150$

Solution:

$$\log_9 150 = y$$

Exercise:

Problem: $b^3 = 45$

Exercise:

Problem: $4^{-3/2} = 0.125$

Solution:

$$\log_4 0.125 = -\frac{3}{2}$$

For the following exercises, sketch the graph of the logarithmic function. Determine the domain, range, and vertical asymptote.

Exercise:

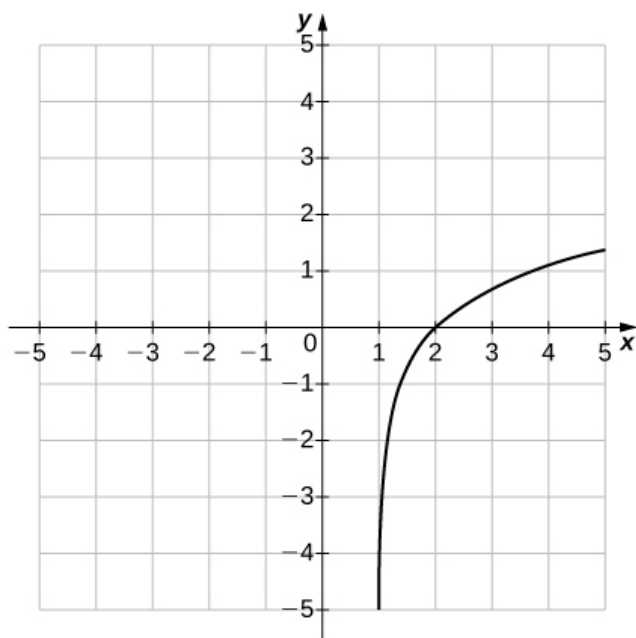
Problem: $f(x) = 3 + \ln x$

Exercise:

Problem: $f(x) = \ln(x - 1)$

Solution:

Domain: $(1, \infty)$, range: $(-\infty, \infty)$, $x = 1$



Exercise:

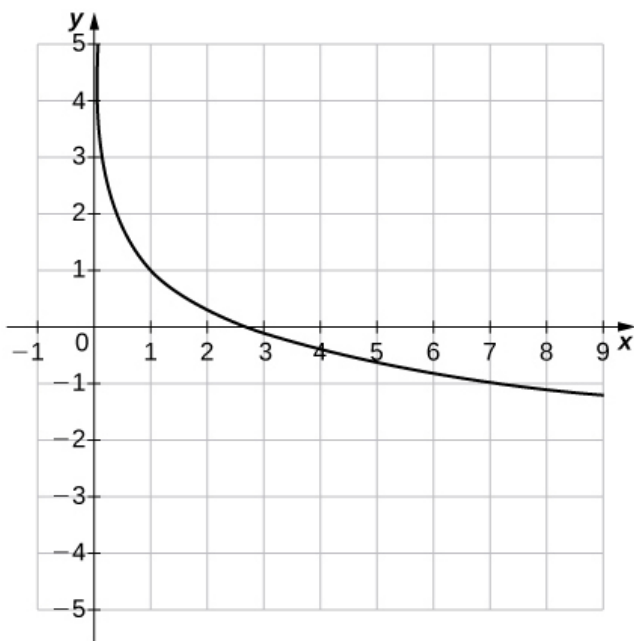
Problem: $f(x) = \ln(-x)$

Exercise:

Problem: $f(x) = 1 - \ln x$

Solution:

Domain: $(0, \infty)$, range: $(-\infty, \infty)$, $x = 0$



Exercise:

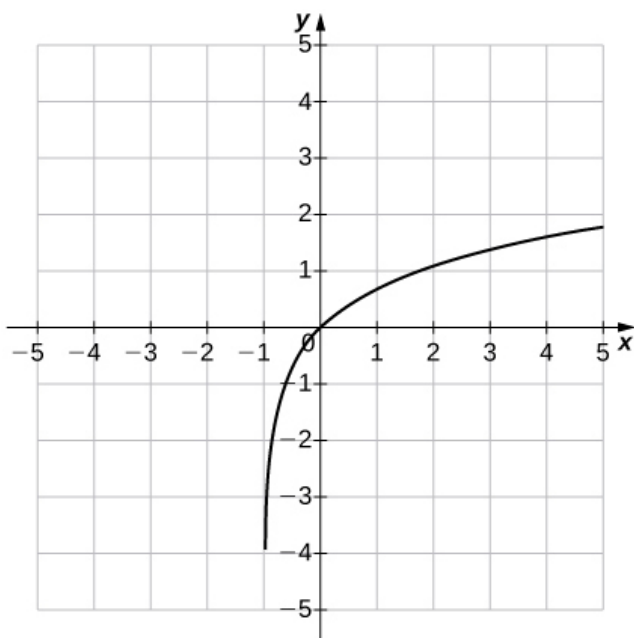
Problem: $f(x) = \log x - 1$

Exercise:

Problem: $f(x) = \ln(x + 1)$

Solution:

Domain: $(-1, \infty)$, range: $(-\infty, \infty)$, $x = -1$



For the following exercises, use properties of logarithms to write the expressions as a sum, difference, and/or product of logarithms.

Exercise:

Problem: $\log x^4 y$

Exercise:

Problem: $\log_3 \frac{9a^3}{b}$

Solution:

$$2 + 3\log_3 a - \log_3 b$$

Exercise:

Problem: $\ln a \sqrt[3]{b}$

Exercise:

Problem: $\log_5 \sqrt{125xy^3}$

Solution:

$$\frac{3}{2} + \frac{1}{2}\log_5 x + \frac{3}{2}\log_5 y$$

Exercise:

Problem: $\log_4 \frac{\sqrt[3]{xy}}{64}$

Exercise:

Problem: $\ln \left(\frac{6}{\sqrt{e^3}} \right)$

Solution:

$$-\frac{3}{2} + \ln 6$$

For the following exercises, solve the exponential equation exactly.

Exercise:

Problem: $5^x = 125$

Exercise:

Problem: $e^{3x} - 15 = 0$

Solution:

$$\frac{\ln 15}{3}$$

Exercise:

Problem: $8^x = 4$

Exercise:

Problem: $4^{x+1} - 32 = 0$

Solution:

$$\frac{3}{2}$$

Exercise:

Problem: $3^{x/14} = \frac{1}{10}$

Exercise:

Problem: $10^x = 7.21$

Solution:

$$\log 7.21$$

Exercise:

Problem: $4 \cdot 2^{3x} - 20 = 0$

Exercise:

Problem: $7^{3x-2} = 11$

Solution:

$$\frac{2}{3} + \frac{\log 11}{3 \log 7}$$

For the following exercises, solve the logarithmic equation exactly, if possible.

Exercise:

Problem: $\log_3 x = 0$

Exercise:

Problem: $\log_5 x = -2$

Solution:

$$x = \frac{1}{25}$$

Exercise:

Problem: $\log_4 (x + 5) = 0$

Exercise:

Problem: $\log (2x - 7) = 0$

Solution:

$$x = 4$$

Exercise:

Problem: $\ln \sqrt{x + 3} = 2$

Exercise:

Problem: $\log_6 (x + 9) + \log_6 x = 2$

Solution:

$$x = 3$$

Exercise:

Problem: $\log_4(x+2) - \log_4(x-1) = 0$

Exercise:

Problem: $\ln x + \ln(x-2) = \ln 4$

Solution:

$$1 + \sqrt{5}$$

For the following exercises, use the change-of-base formula and either base 10 or base e to evaluate the given expressions. Answer in exact form and in approximate form, rounding to four decimal places.

Exercise:

Problem: $\log_5 47$

Exercise:

Problem: $\log_7 82$

Solution:

$$\left(\frac{\log 82}{\log 7} \approx 2.2646 \right)$$

Exercise:

Problem: $\log_6 103$

Exercise:

Problem: $\log_{0.5} 211$

Solution:

$$\left(\frac{\log 211}{\log 0.5} \approx -7.7211 \right)$$

Exercise:

Problem: $\log_2 \pi$

Exercise:

Problem: $\log_{0.2} 0.452$

Solution:

$$\left(\frac{\log 0.452}{\log 0.2} \approx 0.4934 \right)$$

Exercise:

Problem: Rewrite the following expressions in terms of exponentials and simplify.

- a. $2 \cosh (\ln x)$ b. $\cosh 4x + \sinh 4x$ c. $\cosh 2x - \sinh 2x$ d.
 $\ln (\cosh x + \sinh x) + \ln (\cosh x - \sinh x)$

Exercise:

Problem:

[T] The number of bacteria N in a culture after t days can be modeled by the function $N(t) = 1300 \cdot (2)^{t/4}$. Find the number of bacteria present after 15 days.

Solution:

~17,491

Exercise:

Problem:

[T] The demand D (in millions of barrels) for oil in an oil-rich country is given by the function $D(p) = 150 \cdot (2.7)^{-0.25p}$, where p is the price (in dollars) of a barrel of oil. Find the amount of oil demanded (to the nearest million barrels) when the price is between \$15 and \$20.

Exercise:

Problem:

[T] The amount A of a \$100,000 investment paying continuously and compounded for t years is given by $A(t) = 100,000 \cdot e^{0.055t}$. Find the amount A accumulated in 5 years.

Solution:

Approximately \$131,653 is accumulated in 5 years.

Exercise:

Problem:

[T] An investment is compounded monthly, quarterly, or yearly and is given by the function $A = P \left(1 + \frac{j}{n} \right)^{nt}$, where A is the value of the investment at time t , P is the initial principle that was invested, j is the annual interest rate, and n is the number of time the interest is compounded per year. Given a yearly interest rate of 3.5% and an initial principle of \$100,000, find the amount A accumulated in 5 years for interest that is compounded a. daily, b., monthly, c. quarterly, and d. yearly.

Exercise:

Problem:

[T] The concentration of hydrogen ions in a substance is denoted by $[H^+]$, measured in moles per liter. The pH of a substance is defined by the logarithmic function $pH = -\log [H^+]$. This function is used to measure the acidity of a substance. The pH of water is 7. A substance with a pH less than 7 is an acid, whereas one that has a pH of more than 7 is a base.

- Find the pH of the following substances. Round answers to one digit.
- Determine whether the substance is an acid or a base.

- Eggs: $[H^+] = 1.6 \times 10^{-8} \text{ mol/L}$
- Beer: $[H^+] = 3.16 \times 10^{-3} \text{ mol/L}$
- Tomato Juice: $[H^+] = 7.94 \times 10^{-5} \text{ mol/L}$

Solution:

- i. a. pH = 8 b. Base ii. a. pH = 3 b. Acid iii. a. pH = 4 b. Acid

Exercise:**Problem:**

[T] Iodine-131 is a radioactive substance that decays according to the function $Q(t) = Q_0 \cdot e^{-0.08664t}$, where Q_0 is the initial quantity of a sample of the substance and t is in days. Determine how long it takes (to the nearest day) for 95% of a quantity to decay.

Exercise:**Problem:**

[T] According to the World Bank, at the end of 2013 ($t = 0$) the U.S. population was 316 million and was increasing according to the following model:

$P(t) = 316e^{0.0074t}$, where P is measured in millions of people and t is measured in years after 2013.

- Based on this model, what will be the population of the United States in 2020?
- Determine when the U.S. population will be twice what it is in 2013.

Solution:

- a. ~333 million b. 94 years from 2013, or in 2107

Exercise:**Problem:**

[T] The amount A accumulated after 1000 dollars is invested for t years at an interest rate of 4% is modeled by the function $A(t) = 1000(1.04)^t$.

- Find the amount accumulated after 5 years and 10 years.

- b. Determine how long it takes for the original investment to triple.

Exercise:

Problem:

[T] A bacterial colony grown in a lab is known to double in number in 12 hours. Suppose, initially, there are 1000 bacteria present.

- Use the exponential function $Q = Q_0 e^{kt}$ to determine the value k , which is the growth rate of the bacteria. Round to four decimal places.
- Determine approximately how long it takes for 200,000 bacteria to grow.

Solution:

- a. $k \approx 0.0578$ b. ≈ 92 hours

Exercise:

Problem:

[T] The rabbit population on a game reserve doubles every 6 months. Suppose there were 120 rabbits initially.

- Use the exponential function $P = P_0 a^t$ to determine the growth rate constant a . Round to four decimal places.
- Use the function in part a. to determine approximately how long it takes for the rabbit population to reach 3500.

Exercise:

Problem:

[T] The 1906 earthquake in San Francisco had a magnitude of 8.3 on the Richter scale. At the same time, in Japan, an earthquake with magnitude 4.9 caused only minor damage. Approximately how much more energy was released by the San Francisco earthquake than by the Japanese earthquake?

Solution:

The San Francisco earthquake had $10^{3.4}$ or ~ 2512 times more energy than the Japan earthquake.

Chapter Review Exercises

True or False? Justify your answer with a proof or a counterexample.

Exercise:

Problem: A function is always one-to-one.

Exercise:

Problem: $f \circ g = g \circ f$, assuming f and g are functions.

Solution:

False

Exercise:

Problem: A relation that passes the horizontal and vertical line tests is a one-to-one function.

Exercise:

Problem: A relation passing the horizontal line test is a function.

Solution:

False

For the following problems, state the domain and range of the given functions:

$$f = x^2 + 2x - 3, \quad g = \ln(x - 5), \quad h = \frac{1}{x+4}$$

Exercise:

Problem: h

Exercise:

Problem: g

Solution:

Domain: $x > 5$, range: all real numbers

Exercise:

Problem: $h \circ f$

Exercise:

Problem: $g \circ f$

Solution:

Domain: $x > 2$ and $x < -4$, range: all real numbers

Find the degree, y-intercept, and zeros for the following polynomial functions.

Exercise:

Problem: $f(x) = 2x^2 + 9x - 5$

Exercise:

Problem: $f(x) = x^3 + 2x^2 - 2x$

Solution:

Degree of 3, y -intercept: 0, zeros: 0, $\sqrt{3} - 1$, $-1 - \sqrt{3}$

Simplify the following trigonometric expressions.

Exercise:

Problem: $\frac{\tan^2 x}{\sec^2 x} + \cos^2 x$

Exercise:

Problem: $\cos^2 x - \sin^2 x$

Solution:

$$\cos^2 x - \sin^2 x = \cos 2x \text{ or } = \frac{1 - 2\sin^2 x}{2} \text{ or } = \frac{2\cos^2 x - 1}{2}$$

Solve the following trigonometric equations on the interval $\theta = [-2\pi, 2\pi]$ exactly.

Exercise:

Problem: $6\cos^2 x - 3 = 0$

Exercise:

Problem: $\sec^2 x - 2\sec x + 1 = 0$

Solution:

$$0, \pm 2\pi$$

Solve the following logarithmic equations.

Exercise:

Problem: $5^x = 16$

Exercise:

Problem: $\log_2(x + 4) = 3$

Solution:

Are the following functions one-to-one over their domain of existence? Does the function have an inverse? If so, find the inverse $f^{-1}(x)$ of the function. Justify your answer.

Exercise:

Problem: $f(x) = x^2 + 2x + 1$

Exercise:

Problem: $f(x) = \frac{1}{x}$

Solution:

One-to-one; yes, the function has an inverse; inverse: $f^{-1}(x) = \frac{1}{y}$

For the following problems, determine the largest domain on which the function is one-to-one and find the inverse on that domain.

Exercise:

Problem: $f(x) = \sqrt{9 - x}$

Exercise:

Problem: $f(x) = x^2 + 3x + 4$

Solution:

$$x \geq -\frac{3}{2}, f^{-1}(x) = -\frac{3}{2} + \frac{1}{2}\sqrt{4y - 7}$$

Exercise:

Problem:

A car is racing along a circular track with diameter of 1 mi. A trainer standing in the center of the circle marks his progress every 5 sec. After 5 sec, the trainer has to turn 55° to keep up with the car. How fast is the car traveling?

For the following problems, consider a restaurant owner who wants to sell T-shirts advertising his brand. He recalls that there is a fixed cost and variable cost, although he does not remember the values. He does know that the T-shirt printing company charges \$440 for 20 shirts and \$1000 for 100 shirts.

Exercise:

Problem:

- Find the equation $C = f(x)$ that describes the total cost as a function of number of shirts and
- determine how many shirts he must sell to break even if he sells the shirts for \$10 each.

Solution:

a. $C(x) = 300 + 7x$ b. 100 shirts

Exercise:**Problem:**

a. Find the inverse function $x = f^{-1}(C)$ and describe the meaning of this function. b. Determine how many shirts the owner can buy if he has \$8000 to spend.

For the following problems, consider the population of Ocean City, New Jersey, which is cyclical by season.

Exercise:**Problem:**

The population can be modeled by $P(t) = 82.5 - 67.5 \cos[(\pi/6)t]$, where t is time in months ($t = 0$ represents January 1) and P is population (in thousands). During a year, in what intervals is the population less than 20,000? During what intervals is the population more than 140,000?

Solution:

The population is less than 20,000 from December 8 through January 23 and more than 140,000 from May 29 through August 2

Exercise:**Problem:**

In reality, the overall population is most likely increasing or decreasing throughout each year. Let's reformulate the model as $P(t) = 82.5 - 67.5 \cos[(\pi/6)t] + t$, where t is time in months ($t = 0$ represents January 1) and P is population (in thousands). When is the first time the population reaches 200,000?

For the following problems, consider radioactive dating. A human skeleton is found in an archeological dig. Carbon dating is implemented to determine how old the skeleton is by using the equation $y = e^{rt}$, where y is the percentage of radiocarbon still present in the material, t is the number of years passed, and $r = -0.0001210$ is the decay rate of radiocarbon.

Exercise:**Problem:**

If the skeleton is expected to be 2000 years old, what percentage of radiocarbon should be present?

Solution:

78.51%

Exercise:

Problem:

Find the inverse of the carbon-dating equation. What does it mean? If there is 25% radiocarbon, how old is the skeleton?

Glossary

base

the number b in the exponential function $f(x) = b^x$ and the logarithmic function $f(x) = \log_b x$

exponent

the value x in the expression b^x

hyperbolic functions

the functions denoted $\sinh$, $\cosh$, $\tanh$, csch , sech , and coth , which involve certain combinations of e^x and e^{-x}

inverse hyperbolic functions

the inverses of the hyperbolic functions where $\cosh$ and sech are restricted to the domain $[0, \infty)$; each of these functions can be expressed in terms of a composition of the natural logarithm function and an algebraic function

natural exponential function

the function $f(x) = e^x$

natural logarithm

the function $\ln x = \log_e x$

number e

as m gets larger, the quantity $(1 + (1/m))^m$ gets closer to some real number; we define that real number to be e ; the value of e is approximately 2.718282

Derivatives of Trigonometric Functions

- Find the derivatives of the sine and cosine function.
- Find the derivatives of the standard trigonometric functions.
- Calculate the higher-order derivatives of the sine and cosine.

One of the most important types of motion in physics is simple harmonic motion, which is associated with such systems as an object with mass oscillating on a spring. Simple harmonic motion can be described by using either sine or cosine functions. In this section we expand our knowledge of derivative formulas to include derivatives of these and other trigonometric functions. We begin with the derivatives of the sine and cosine functions and then use them to obtain formulas for the derivatives of the remaining four trigonometric functions. Being able to calculate the derivatives of the sine and cosine functions will enable us to find the velocity and acceleration of simple harmonic motion.

Derivatives of the Sine and Cosine Functions

We begin our exploration of the derivative for the sine function by using the formula to make a reasonable guess at its derivative. Recall that for a function $f(x)$,

Equation:

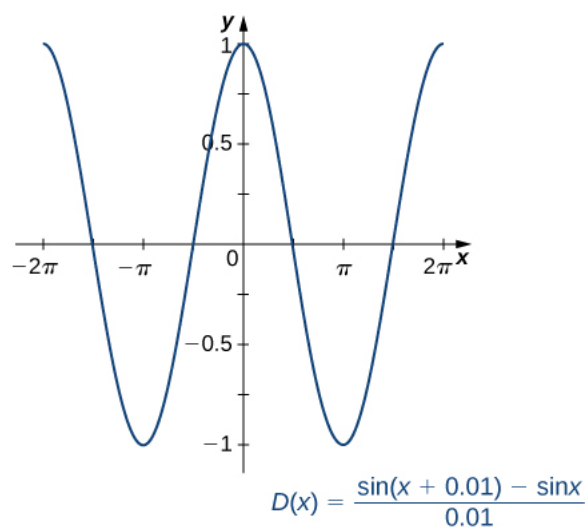
$$f'(x) = \lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h}.$$

Consequently, for values of h very close to 0, $f'(x) \approx \frac{f(x+h) - f(x)}{h}$. We see that by using $h = 0.01$,

Equation:

$$\frac{d}{dx}(\sin x) \approx \frac{\sin(x + 0.01) - \sin x}{0.01}$$

By setting $D(x) = \frac{\sin(x+0.01) - \sin x}{0.01}$ and using a graphing utility, we can get a graph of an approximation to the derivative of $\sin x$ ([link](#)).



The graph of the function $D(x)$ looks a lot like a cosine curve.

Upon inspection, the graph of $D(x)$ appears to be very close to the graph of the cosine function. Indeed, we will show that

Equation:

$$\frac{d}{dx}(\sin x) = \cos x.$$

If we were to follow the same steps to approximate the derivative of the cosine function, we would find that

Equation:

$$\frac{d}{dx}(\cos x) = -\sin x.$$

Note:

The Derivatives of $\sin x$ and $\cos x$

The derivative of the sine function is the cosine and the derivative of the cosine function is the negative sine.

Equation:

$$\frac{d}{dx}(\sin x) = \cos x$$

Equation:

$$\frac{d}{dx}(\cos x) = -\sin x$$

Proof

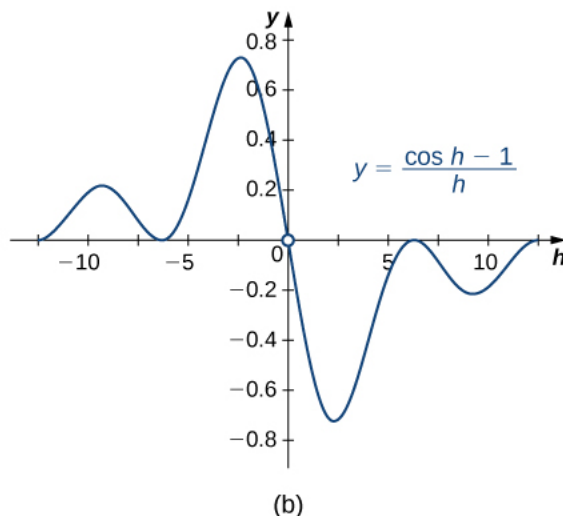
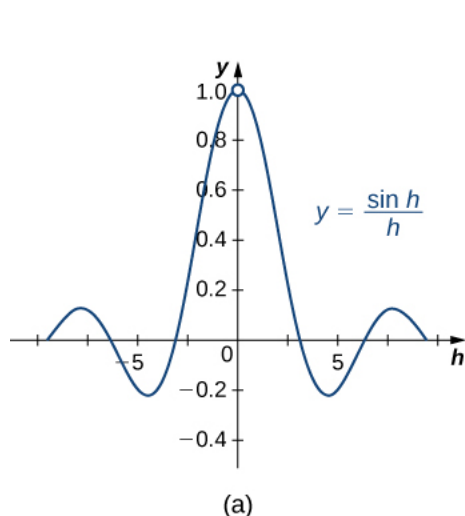
Because the proofs for $\frac{d}{dx}(\sin x) = \cos x$ and $\frac{d}{dx}(\cos x) = -\sin x$ use similar techniques, we provide only the proof for $\frac{d}{dx}(\sin x) = \cos x$. Before beginning, recall two important trigonometric limits we learned in

[Introduction to Limits](#):

Equation:

$$\lim_{h \rightarrow 0} \frac{\sin h}{h} = 1 \text{ and } \lim_{h \rightarrow 0} \frac{\cos h - 1}{h} = 0.$$

The graphs of $y = \frac{(\sin h)}{h}$ and $y = \frac{(\cos h - 1)}{h}$ are shown in [\[link\]](#).



These graphs show two important limits needed to establish the derivative formulas for the sine and cosine functions.

We also recall the following trigonometric identity for the sine of the sum of two angles:

Equation:

$$\sin(x + h) = \sin x \cos h + \cos x \sin h.$$

Now that we have gathered all the necessary equations and identities, we proceed with the proof.

Equation:

$$\begin{aligned} \frac{d}{dx} \sin x &= \lim_{h \rightarrow 0} \frac{\sin(x+h) - \sin x}{h} \\ &= \lim_{h \rightarrow 0} \frac{\sin x \cos h + \cos x \sin h - \sin x}{h} \\ &= \lim_{h \rightarrow 0} \left(\frac{\sin x \cos h - \sin x}{h} + \frac{\cos x \sin h}{h} \right) \\ &= \lim_{h \rightarrow 0} \left(\sin x \left(\frac{\cos h - 1}{h} \right) + \cos x \left(\frac{\sin h}{h} \right) \right) \\ &= \sin x \cdot 0 + \cos x \cdot 1 \\ &= \cos x \end{aligned}$$

Apply the definition of the derivative.

Use trig identity for the sine of the sum of two angles

Regroup.

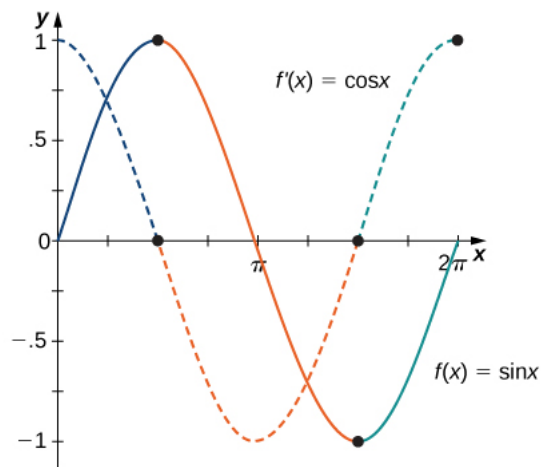
Factor out $\sin x$ and $\cos x$.

Apply trig limit formulas.

Simplify.

□

[\[link\]](#) shows the relationship between the graph of $f(x) = \sin x$ and its derivative $f'(x) = \cos x$. Notice that at the points where $f(x) = \sin x$ has a horizontal tangent, its derivative $f'(x) = \cos x$ takes on the value zero. We also see that where $f(x) = \sin x$ is increasing, $f'(x) = \cos x > 0$ and where $f(x) = \sin x$ is decreasing, $f'(x) = \cos x < 0$.



Where $f(x)$ has a maximum or a minimum, $f'(x) = 0$
that is, $f'(x) = 0$ where $f(x)$ has a horizontal tangent.

These points are noted with dots on the graphs.

Example:

Exercise:

Problem:

Differentiating a Function Containing $\sin x$

Find the derivative of $f(x) = 5x^3 \sin x$.

Solution:

Using the product rule, we have

Equation:

$$\begin{aligned} f'(x) &= \frac{d}{dx}(5x^3) \cdot \sin x + \frac{d}{dx}(\sin x) \cdot 5x^3 \\ &= 15x^2 \cdot \sin x + \cos x \cdot 5x^3. \end{aligned}$$

After simplifying, we obtain

Equation:

$$f'(x) = 15x^2 \sin x + 5x^3 \cos x.$$

Note:

Exercise:

Problem: Find the derivative of $f(x) = \sin x \cos x$.

Solution:

$$f'(x) = \cos^2 x - \sin^2 x$$

Hint

Don't forget to use the product rule.

Example:

Exercise:

Problem:

Finding the Derivative of a Function Containing $\cos x$

Find the derivative of $g(x) = \frac{\cos x}{4x^2}$.

Solution:

By applying the quotient rule, we have

Equation:

$$g'(x) = \frac{(-\sin x)4x^2 - 8x(\cos x)}{(4x^2)^2}.$$

Simplifying, we obtain

Equation:

$$\begin{aligned} g'(x) &= \frac{-4x^2 \sin x - 8x \cos x}{16x^4} \\ &= \frac{-x \sin x - 2 \cos x}{4x^3}. \end{aligned}$$

Note:

Exercise:

Problem: Find the derivative of $f(x) = \frac{x}{\cos x}$.

Solution:

$$\frac{\cos x + x \sin x}{\cos^2 x}$$

Hint

Use the quotient rule.

Example:

Exercise:

Problem:

An Application to Velocity

A particle moves along a coordinate axis in such a way that its position at time t is given by $s(t) = 2 \sin t - t$ for $0 \leq t \leq 2\pi$. At what times is the particle at rest?

Solution:

To determine when the particle is at rest, set $s'(t) = v(t) = 0$. Begin by finding $s'(t)$. We obtain

Equation:

$$s'(t) = 2\cos t - 1,$$

so we must solve

Equation:

$$2\cos t - 1 = 0 \text{ for } 0 \leq t \leq 2\pi.$$

The solutions to this equation are $t = \frac{\pi}{3}$ and $t = \frac{5\pi}{3}$. Thus the particle is at rest at times $t = \frac{\pi}{3}$ and $t = \frac{5\pi}{3}$.

Note:**Exercise:****Problem:**

A particle moves along a coordinate axis. Its position at time t is given by $s(t) = \sqrt{3}t + 2\cos t$ for $0 \leq t \leq 2\pi$. At what times is the particle at rest?

Solution:

$$t = \frac{\pi}{3}, t = \frac{2\pi}{3}$$

Hint

Use the previous example as a guide.

Derivatives of Other Trigonometric Functions

Since the remaining four trigonometric functions may be expressed as quotients involving sine, cosine, or both, we can use the quotient rule to find formulas for their derivatives.

Example:**Exercise:****Problem:****The Derivative of the Tangent Function**

Find the derivative of $f(x) = \tan x$.

Solution:

Start by expressing $\tan x$ as the quotient of $\sin x$ and $\cos x$:

Equation:

$$f(x) = \tan x = \frac{\sin x}{\cos x}.$$

Now apply the quotient rule to obtain

Equation:

$$f'(x) = \frac{\cos x \cos x - (-\sin x)\sin x}{(\cos x)^2}.$$

Simplifying, we obtain

Equation:

$$f'(x) = \frac{\cos^2 x + \sin^2 x}{\cos^2 x}.$$

Recognizing that $\cos^2 x + \sin^2 x = 1$, by the Pythagorean theorem, we now have

Equation:

$$f'(x) = \frac{1}{\cos^2 x}.$$

Finally, use the identity $\sec x = \frac{1}{\cos x}$ to obtain

Equation:

$$f'(x) = \sec^2 x.$$

Note:

Exercise:

Problem: Find the derivative of $f(x) = \cot x$.

Solution:

$$f'(x) = -\csc^2 x$$

Hint

Rewrite $\cot x$ as $\frac{\cos x}{\sin x}$ and use the quotient rule.

The derivatives of the remaining trigonometric functions may be obtained by using similar techniques. We provide these formulas in the following theorem.

Note:

Derivatives of $\tan x$, $\cot x$, $\sec x$, and $\csc x$

The derivatives of the remaining trigonometric functions are as follows:

Equation:

$$\frac{d}{dx}(\tan x) = \sec^2 x$$

Equation:

$$\frac{d}{dx}(\cot x) = -\csc^2 x$$

Equation:

$$\frac{d}{dx}(\sec x) = \sec x \tan x$$

Equation:

$$\frac{d}{dx}(\csc x) = -\csc x \cot x.$$

Example:

Exercise:

Problem:

Finding the Equation of a Tangent Line

Find the equation of a line tangent to the graph of $f(x) = \cot x$ at $x = \frac{\pi}{4}$.

Solution:

To find the equation of the tangent line, we need a point and a slope at that point. To find the point, compute

Equation:

$$f\left(\frac{\pi}{4}\right) = \cot \frac{\pi}{4} = 1.$$

Thus the tangent line passes through the point $\left(\frac{\pi}{4}, 1\right)$. Next, find the slope by finding the derivative of $f(x) = \cot x$ and evaluating it at $\frac{\pi}{4}$:

Equation:

$$f'(x) = -\csc^2 x \text{ and } f'\left(\frac{\pi}{4}\right) = -\csc^2\left(\frac{\pi}{4}\right) = -2.$$

Using the point-slope equation of the line, we obtain

Equation:

$$y - 1 = -2\left(x - \frac{\pi}{4}\right)$$

or equivalently,

Equation:

$$y = -2x + 1 + \frac{\pi}{2}.$$

Example:

Exercise:

Problem:

Finding the Derivative of Trigonometric Functions

Find the derivative of $f(x) = \csc x + x \tan x$.

Solution:

To find this derivative, we must use both the sum rule and the product rule. Using the sum rule, we find

Equation:

$$f'(x) = \frac{d}{dx}(\csc x) + \frac{d}{dx}(x \tan x).$$

In the first term, $\frac{d}{dx}(\csc x) = -\csc x \cot x$, and by applying the product rule to the second term we obtain

Equation:

$$\frac{d}{dx}(x \tan x) = (1)(\tan x) + (\sec^2 x)(x).$$

Therefore, we have

Equation:

$$f'(x) = -\csc x \cot x + \tan x + x \sec^2 x.$$

Note:

Exercise:

Problem: Find the derivative of $f(x) = 2 \tan x - 3 \cot x$.

Solution:

$$f'(x) = 2 \sec^2 x + 3 \csc^2 x$$

Hint

Use the rule for differentiating a constant multiple and the rule for differentiating a difference of two functions.

Note:

Exercise:

Problem: Find the slope of the line tangent to the graph of $f(x) = \tan x$ at $x = \frac{\pi}{6}$.

Solution:

$$\frac{4}{3}$$

Hint

Evaluate the derivative at $x = \frac{\pi}{6}$.

Higher-Order Derivatives

The higher-order derivatives of $\sin x$ and $\cos x$ follow a repeating pattern. By following the pattern, we can find any higher-order derivative of $\sin x$ and $\cos x$.

Example:

Exercise:

Problem:

Finding Higher-Order Derivatives of $y = \sin x$

Find the first four derivatives of $y = \sin x$.

Solution:

Each step in the chain is straightforward:

Equation:

$$\begin{aligned}y &= \sin x \\ \frac{dy}{dx} &= \cos x \\ \frac{d^2y}{dx^2} &= -\sin x \\ \frac{d^3y}{dx^3} &= -\cos x \\ \frac{d^4y}{dx^4} &= \sin x.\end{aligned}$$

Analysis

Once we recognize the pattern of derivatives, we can find any higher-order derivative by determining the step in the pattern to which it corresponds. For example, every fourth derivative of $\sin x$ equals $\sin x$, so

Equation:

$$\begin{aligned}\frac{d^4}{dx^4}(\sin x) &= \frac{d^8}{dx^8}(\sin x) = \frac{d^{12}}{dx^{12}}(\sin x) = \dots = \frac{d^{4n}}{dx^{4n}}(\sin x) = \sin x \\ \frac{d^5}{dx^5}(\sin x) &= \frac{d^9}{dx^9}(\sin x) = \frac{d^{13}}{dx^{13}}(\sin x) = \dots = \frac{d^{4n+1}}{dx^{4n+1}}(\sin x) = \cos x.\end{aligned}$$

Note:

Exercise:

Problem: For $y = \cos x$, find $\frac{d^4y}{dx^4}$.

Solution:

$$\cos x$$

Hint

See the previous example.

Example:**Exercise:****Problem:**

Using the Pattern for Higher-Order Derivatives of $y = \sin x$

Find $\frac{d^{74}}{dx^{74}}(\sin x)$.

Solution:

We can see right away that for the 74th derivative of $\sin x$, $74 = 4(18) + 2$, so

Equation:

$$\frac{d^{74}}{dx^{74}}(\sin x) = \frac{d^{72+2}}{dx^{72+2}}(\sin x) = \frac{d^2}{dx^2}(\sin x) = -\sin x.$$

Note:**Exercise:**

Problem: For $y = \sin x$, find $\frac{d^{59}}{dx^{59}}(\sin x)$.

Solution:

$$-\cos x$$

Hint

$$\frac{d^{59}}{dx^{59}}(\sin x) = \frac{d^{4 \cdot 14 + 3}}{dx^{4 \cdot 14 + 3}}(\sin x)$$

Example:**Exercise:****Problem:**

An Application to Acceleration

A particle moves along a coordinate axis in such a way that its position at time t is given by $s(t) = 2 - \sin t$. Find $v(\pi/4)$ and $a(\pi/4)$. Compare these values and decide whether the particle is speeding up or slowing down.

Solution:

First find $v(t) = s'(t)$:

Equation:

$$v(t) = s'(t) = -\cos t.$$

Thus,

Equation:

$$v\left(\frac{\pi}{4}\right) = -\frac{1}{\sqrt{2}}.$$

Next, find $a(t) = v'(t)$. Thus, $a(t) = v'(t) = \sin t$ and we have

Equation:

$$a\left(\frac{\pi}{4}\right) = \frac{1}{\sqrt{2}}.$$

Since $v\left(\frac{\pi}{4}\right) = -\frac{1}{\sqrt{2}} < 0$ and $a\left(\frac{\pi}{4}\right) = \frac{1}{\sqrt{2}} > 0$, we see that velocity and acceleration are acting in opposite directions; that is, the object is being accelerated in the direction opposite to the direction in which it is travelling. Consequently, the particle is slowing down.

Note:

Exercise:

Problem:

A block attached to a spring is moving vertically. Its position at time t is given by $s(t) = 2\sin t$. Find $v\left(\frac{5\pi}{6}\right)$ and $a\left(\frac{5\pi}{6}\right)$. Compare these values and decide whether the block is speeding up or slowing down.

Solution:

$v\left(\frac{5\pi}{6}\right) = -\sqrt{3} < 0$ and $a\left(\frac{5\pi}{6}\right) = -1 < 0$. The block is speeding up.

Hint

Use [\[link\]](#) as a guide.

Key Concepts

- We can find the derivatives of $\sin x$ and $\cos x$ by using the definition of derivative and the limit formulas found earlier. The results are

Equation:

$$\frac{d}{dx} \sin x = \cos x \quad \frac{d}{dx} \cos x = -\sin x.$$

- With these two formulas, we can determine the derivatives of all six basic trigonometric functions.

Key Equations

- Derivative of sine function**

$$\frac{d}{dx} (\sin x) = \cos x$$

- Derivative of cosine function**

$$\frac{d}{dx} (\cos x) = -\sin x$$

- Derivative of tangent function**

$$\frac{d}{dx} (\tan x) = \sec^2 x$$

- Derivative of cotangent function**

$$\frac{d}{dx} (\cot x) = -\csc^2 x$$

- Derivative of secant function**

$$\frac{d}{dx} (\sec x) = \sec x \tan x$$

- **Derivative of cosecant function**

$$\frac{d}{dx}(\csc x) = -\csc x \cot x$$

For the following exercises, find $\frac{dy}{dx}$ for the given functions.

Exercise:

Problem: $y = x^2 - \sec x + 1$

Solution:

$$\frac{dy}{dx} = 2x - \sec x \tan x$$

Exercise:

Problem: $y = 3 \csc x + \frac{5}{x}$

Exercise:

Problem: $y = x^2 \cot x$

Solution:

$$\frac{dy}{dx} = 2x \cot x - x^2 \csc^2 x$$

Exercise:

Problem: $y = x - x^3 \sin x$

Exercise:

Problem: $y = \frac{\sec x}{x}$

Solution:

$$\frac{dy}{dx} = \frac{x \sec x \tan x - \sec x}{x^2}$$

Exercise:

Problem: $y = \sin x \tan x$

Exercise:

Problem: $y = (x + \cos x)(1 - \sin x)$

Solution:

$$\frac{dy}{dx} = (1 - \sin x)(1 - \sin x) - \cos x(x + \cos x)$$

Exercise:

Problem: $y = \frac{\tan x}{1 - \sec x}$

Exercise:

Problem: $y = \frac{1 - \cot x}{1 + \cot x}$

Solution:

$$\frac{dy}{dx} = \frac{2\csc^2 x}{(1+\cot x)^2}$$

Exercise:

Problem: $y = \cos x (1 + \csc x)$

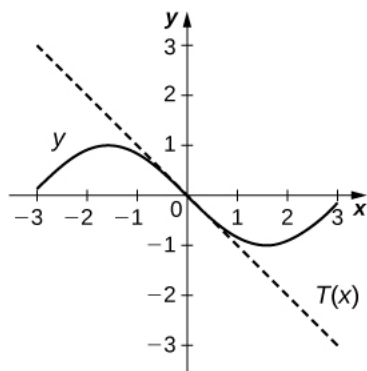
For the following exercises, find the equation of the tangent line to each of the given functions at the indicated values of x . Then use a calculator to graph both the function and the tangent line to ensure the equation for the tangent line is correct.

Exercise:

Problem: [T] $f(x) = -\sin x, x = 0$

Solution:

$$y = -x$$



Exercise:

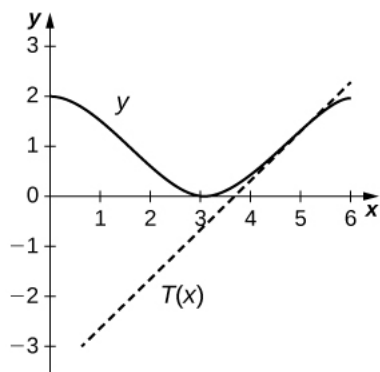
Problem: [T] $f(x) = \csc x, x = \frac{\pi}{2}$

Exercise:

Problem: [T] $f(x) = 1 + \cos x, x = \frac{3\pi}{2}$

Solution:

$$y = x + \frac{2-3\pi}{2}$$



Exercise:

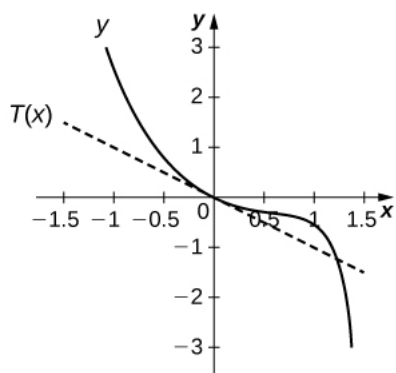
Problem: [T] $f(x) = \sec x$, $x = \frac{\pi}{4}$

Exercise:

Problem: [T] $f(x) = x^2 - \tan x$, $x = 0$

Solution:

$$y = -x$$



Exercise:

Problem: [T] $f(x) = 5 \cot x$, $x = \frac{\pi}{4}$

For the following exercises, find $\frac{d^2y}{dx^2}$ for the given functions.

Exercise:

Problem: $y = x \sin x - \cos x$

Solution:

$$3 \cos x - x \sin x$$

Exercise:

Problem: $y = \sin x \cos x$

Exercise:

Problem: $y = x - \frac{1}{2} \sin x$

Solution:

$$\frac{1}{2} \sin x$$

Exercise:

Problem: $y = \frac{1}{x} + \tan x$

Exercise:

Problem: $y = 2 \csc x$

Solution:

$$2 \csc x (\csc^2 x + \cot^2 x)$$

Exercise:

Problem: $y = \sec^2 x$

Exercise:

Problem: Find all x values on the graph of $f(x) = -3 \sin x \cos x$ where the tangent line is horizontal.

Solution:

$$\frac{(2n+1)\pi}{4}, \text{ where } n \text{ is an integer}$$

Exercise:

Problem:

Find all x values on the graph of $f(x) = x - 2 \cos x$ for $0 < x < 2\pi$ where the tangent line has slope 2.

Exercise:

Problem:

Let $f(x) = \cot x$. Determine the points on the graph of f for $0 < x < 2\pi$ where the tangent line(s) is (are) parallel to the line $y = -2x$.

Solution:

$$\left(\frac{\pi}{4}, 1\right), \left(\frac{3\pi}{4}, -1\right)$$

Exercise:

Problem:

[T] A mass on a spring bounces up and down in simple harmonic motion, modeled by the function $s(t) = -6 \cos t$ where s is measured in inches and t is measured in seconds. Find the rate at which the spring is oscillating at $t = 5$ s.

Exercise:**Problem:**

Let the position of a swinging pendulum in simple harmonic motion be given by $s(t) = a \cos t + b \sin t$ where a and b are constants, t measures time in seconds, and s measures position in centimeters. If the position is 0 cm and the velocity is 3 cm/s when $t = 0$, find the values of a and b .

Solution:

$$a = 0, b = 3$$

Exercise:**Problem:**

After a diver jumps off a diving board, the edge of the board oscillates with position given by $s(t) = -5 \cos t$ cm at t seconds after the jump.

- Sketch one period of the position function for $t \geq 0$.
- Find the velocity function.
- Sketch one period of the velocity function for $t \geq 0$.
- Determine the times when the velocity is 0 over one period.
- Find the acceleration function.
- Sketch one period of the acceleration function for $t \geq 0$.

Exercise:**Problem:**

The number of hamburgers sold at a fast-food restaurant in Pasadena, California, is given by $y = 10 + 5 \sin x$ where y is the number of hamburgers sold and x represents the number of hours after the restaurant opened at 11 a.m. until 11 p.m., when the store closes. Find y' and determine the intervals where the number of burgers being sold is increasing.

Solution:

$$y' = 5 \cos(x), \text{ increasing on } \left(0, \frac{\pi}{2}\right), \left(\frac{3\pi}{2}, \frac{5\pi}{2}\right), \text{ and } \left(\frac{7\pi}{2}, 12\right)$$

Exercise:**Problem:**

[T] The amount of rainfall per month in Phoenix, Arizona, can be approximated by $y(t) = 0.5 + 0.3 \cos t$, where t is months since January. Find y' and use a calculator to determine the intervals where the amount of rain falling is decreasing.

For the following exercises, use the quotient rule to derive the given equations.

Exercise:

Problem: $\frac{d}{dx}(\cot x) = -\csc^2 x$

Exercise:

Problem: $\frac{d}{dx}(\sec x) = \sec x \tan x$

Exercise:

Problem: $\frac{d}{dx}(\csc x) = -\csc x \cot x$

Exercise:

Problem: Use the definition of derivative and the identity

$$\cos (x+h)=\cos x \cos h-\sin x \sin h \text { to prove that } \frac{d(\cos x)}{d x}=-\sin x .$$

For the following exercises, find the requested higher-order derivative for the given functions.

Exercise:

Problem: $\frac{d^3 y}{d x^3}$ of $y=3 \cos x$

Solution:

$$3 \sin x$$

Exercise:

Problem: $\frac{d^2 y}{d x^2}$ of $y=3 \sin x+x^2 \cos x$

Exercise:

Problem: $\frac{d^4 y}{d x^4}$ of $y=5 \cos x$

Solution:

$$5 \cos x$$

Exercise:

Problem: $\frac{d^2 y}{d x^2}$ of $y=\sec x+\cot x$

Exercise:

Problem: $\frac{d^3 y}{d x^3}$ of $y=x^{10}-\sec x$

Solution:

$$720 x^7-5 \tan (x) \sec ^3(x)-\tan ^3(x) \sec (x)$$

Derivatives of Inverse Functions

- Calculate the derivative of an inverse function.
- Recognize the derivatives of the standard inverse trigonometric functions.

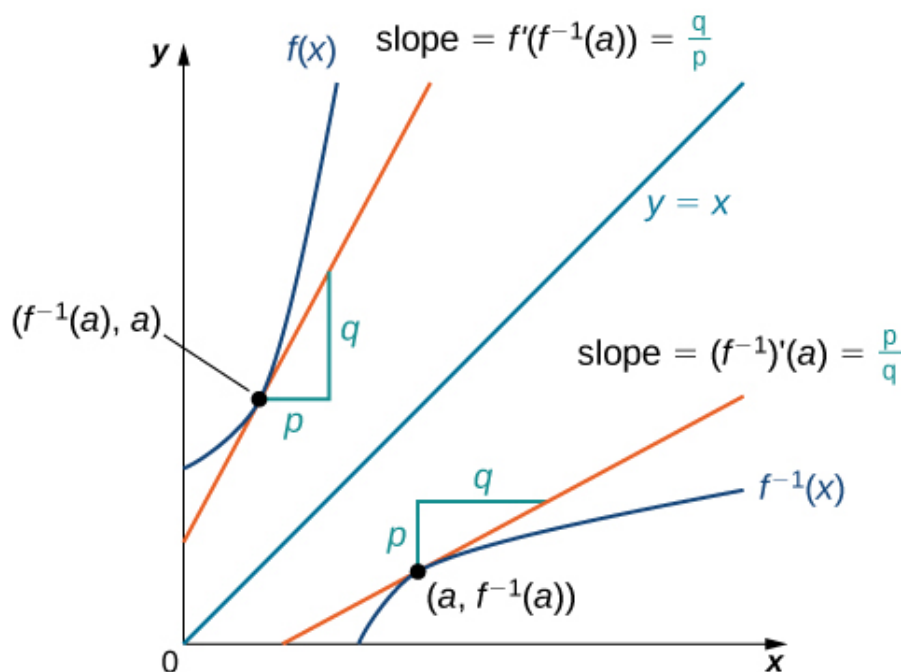
In this section we explore the relationship between the derivative of a function and the derivative of its inverse. For functions whose derivatives we already know, we can use this relationship to find derivatives of inverses without having to use the limit definition of the derivative. In particular, we will apply the formula for derivatives of inverse functions to trigonometric functions. This formula may also be used to extend the power rule to rational exponents.

The Derivative of an Inverse Function

We begin by considering a function and its inverse. If $f(x)$ is both invertible and differentiable, it seems reasonable that the inverse of $f(x)$ is also differentiable. [\[link\]](#) shows the relationship between a function $f(x)$ and its inverse $f^{-1}(x)$. Look at the point $(a, f^{-1}(a))$ on the graph of $f^{-1}(x)$ having a tangent line with a slope of $(f^{-1})'(a) = \frac{p}{q}$. This point corresponds to a point $(f^{-1}(a), a)$ on the graph of $f(x)$ having a tangent line with a slope of $f'(f^{-1}(a)) = \frac{q}{p}$. Thus, if $f^{-1}(x)$ is differentiable at a , then it must be the case that

Equation:

$$(f^{-1})'(a) = \frac{1}{f'(f^{-1}(a))}.$$



The tangent lines of a function and its inverse are related; so, too, are the derivatives of these functions.

We may also derive the formula for the derivative of the inverse by first recalling that $x = f(f^{-1}(x))$. Then by differentiating both sides of this equation (using the chain rule on the right), we obtain

Equation:

$$1 = f'(f^{-1}(x)) (f^{-1})'(x).$$

Solving for $(f^{-1})'(x)$, we obtain

Equation:

$$(f^{-1})'(x) = \frac{1}{f'(f^{-1}(x))}.$$

We summarize this result in the following theorem.

Note:**Inverse Function Theorem**

Let $f(x)$ be a function that is both invertible and differentiable. Let $y = f^{-1}(x)$ be the inverse of $f(x)$. For all x satisfying $f'(f^{-1}(x)) \neq 0$,

Equation:

$$\frac{dy}{dx} = \frac{d}{dx}(f^{-1}(x)) = (f^{-1})'(x) = \frac{1}{f'(f^{-1}(x))}.$$

Alternatively, if $y = g(x)$ is the inverse of $f(x)$, then

Equation:

$$g'(x) = \frac{1}{f'(g(x))}.$$

Example:**Exercise:****Problem:****Applying the Inverse Function Theorem**

Use the inverse function theorem to find the derivative of $g(x) = \frac{x+2}{x}$.

Compare the resulting derivative to that obtained by differentiating the function directly.

Solution:

The inverse of $g(x) = \frac{x+2}{x}$ is $f(x) = \frac{2}{x-1}$. Since $g'(x) = \frac{1}{f'(g(x))}$, begin by finding $f'(x)$. Thus,

Equation:

$$f'(x) = \frac{-2}{(x-1)^2} \text{ and } f'(g(x)) = \frac{-2}{(g(x)-1)^2} = \frac{-2}{\left(\frac{x+2}{x}-1\right)^2} = -\frac{x^2}{2}.$$

Finally,

Equation:

$$g'(x) = \frac{1}{f'(g(x))} = -\frac{2}{x^2}.$$

We can verify that this is the correct derivative by applying the quotient rule to $g(x)$ to obtain

Equation:

$$g'(x) = -\frac{2}{x^2}.$$

Note:

Exercise:

Problem:

Use the inverse function theorem to find the derivative of $g(x) = \frac{1}{x+2}$.
Compare the result obtained by differentiating $g(x)$ directly.

Solution:

$$g'(x) = -\frac{1}{(x+2)^2}$$

Hint

Use the preceding example as a guide.

Example:

Exercise:

Problem:

Applying the Inverse Function Theorem

Use the inverse function theorem to find the derivative of $g(x) = \sqrt[3]{x}$.

Solution:

The function $g(x) = \sqrt[3]{x}$ is the inverse of the function $f(x) = x^3$. Since $g'(x) = \frac{1}{f'(g(x))}$, begin by finding $f'(x)$. Thus,

Equation:

$$f'(x) = 3x^2 \text{ and } f'(g(x)) = 3(\sqrt[3]{x})^2 = 3x^{2/3}.$$

Finally,

Equation:

$$g'(x) = \frac{1}{3x^{2/3}} = \frac{1}{3}x^{-2/3}.$$

Note:

Exercise:

Problem:

Find the derivative of $g(x) = \sqrt[5]{x}$ by applying the inverse function theorem.

Solution:

$$g(x) = \frac{1}{5}x^{-4/5}$$

Hint

$g(x)$ is the inverse of $f(x) = x^5$.

From the previous example, we see that we can use the inverse function theorem to extend the power rule to exponents of the form $\frac{1}{n}$, where n is a positive integer. This extension will ultimately allow us to differentiate x^q , where q is any rational number.

Note:**Extending the Power Rule to Rational Exponents**

The power rule may be extended to rational exponents. That is, if n is a positive integer, then

Equation:

$$\frac{d}{dx} \left(x^{1/n} \right) = \frac{1}{n} x^{(1/n)-1}.$$

Also, if n is a positive integer and m is an arbitrary integer, then

Equation:

$$\frac{d}{dx} \left(x^{m/n} \right) = \frac{m}{n} x^{(m/n)-1}.$$

Proof

The function $g(x) = x^{1/n}$ is the inverse of the function $f(x) = x^n$. Since $g'(x) = \frac{1}{f'(g(x))}$, begin by finding $f'(x)$. Thus,

Equation:

$$f'(x) = nx^{n-1} \text{ and } f'(g(x)) = n(x^{1/n})^{n-1} = nx^{(n-1)/n}.$$

Finally,

Equation:

$$g'(x) = \frac{1}{nx^{(n-1)/n}} = \frac{1}{n} x^{(1-n)/n} = \frac{1}{n} x^{(1/n)-1}.$$

To differentiate $x^{m/n}$ we must rewrite it as $(x^{1/n})^m$ and apply the chain rule.

Thus,

Equation:

$$\frac{d}{dx} \left(x^{m/n} \right) = \frac{d}{dx} \left(\left(x^{1/n} \right)^m \right) = m \left(x^{1/n} \right)^{m-1} \cdot \frac{1}{n} x^{(1/n)-1} = \frac{m}{n} x^{(m/n)-1}.$$



Example:

Exercise:

Problem:

Applying the Power Rule to a Rational Power

Find the equation of the line tangent to the graph of $y = x^{2/3}$ at $x = 8$.

Solution:

First find $\frac{dy}{dx}$ and evaluate it at $x = 8$. Since

Equation:

$$\frac{dy}{dx} = \frac{2}{3}x^{-1/3} \text{ and } \left. \frac{dy}{dx} \right|_{x=8} = \frac{1}{3}$$

the slope of the tangent line to the graph at $x = 8$ is $\frac{1}{3}$.

Substituting $x = 8$ into the original function, we obtain $y = 4$. Thus, the tangent line passes through the point $(8, 4)$. Substituting into the point-slope formula for a line, we obtain the tangent line

Equation:

$$y = \frac{1}{3}x + \frac{4}{3}.$$

Note:

Exercise:

Problem: Find the derivative of $s(t) = \sqrt{2t + 1}$.

Solution:

$$s'(t) = (2t + 1)^{-1/2}$$

Hint

Use the chain rule.

Derivatives of Inverse Trigonometric Functions

We now turn our attention to finding derivatives of inverse trigonometric functions. These derivatives will prove invaluable in the study of integration later in this text. The derivatives of inverse trigonometric functions are quite surprising in that their derivatives are actually algebraic functions. Previously, derivatives of algebraic functions have proven to be algebraic functions and derivatives of trigonometric functions have been shown to be trigonometric functions. Here, for the first time, we see that the derivative of a function need not be of the same type as the original function.

Example:**Exercise:****Problem:****Derivative of the Inverse Sine Function**

Use the inverse function theorem to find the derivative of $g(x) = \sin^{-1}x$.

Solution:

Since for x in the interval $[-\frac{\pi}{2}, \frac{\pi}{2}]$, $f(x) = \sin x$ is the inverse of $g(x) = \sin^{-1}x$, begin by finding $f'(x)$. Since

Equation:

$$f'(x) = \cos x \text{ and } f'(g(x)) = \cos(\sin^{-1}x) = \sqrt{1 - x^2},$$

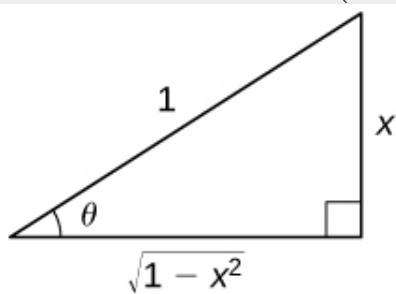
we see that

Equation:

$$g'(x) = \frac{d}{dx} (\sin^{-1} x) = \frac{1}{f'(g(x))} = \frac{1}{\sqrt{1-x^2}}.$$

Analysis

To see that $\cos (\sin^{-1} x) = \sqrt{1-x^2}$, consider the following argument. Set $\sin^{-1} x = \theta$. In this case, $\sin \theta = x$ where $-\frac{\pi}{2} \leq \theta \leq \frac{\pi}{2}$. We begin by considering the case where $0 < \theta < \frac{\pi}{2}$. Since θ is an acute angle, we may construct a right triangle having acute angle θ , a hypotenuse of length 1 and the side opposite angle θ having length x . From the Pythagorean theorem, the side adjacent to angle θ has length $\sqrt{1-x^2}$. This triangle is shown in [\[link\]](#). Using the triangle, we see that $\cos (\sin^{-1} x) = \cos \theta = \sqrt{1-x^2}$.



Using a right triangle having acute angle θ , a hypotenuse of length 1, and the side opposite angle θ having length x , we can see that $\cos (\sin^{-1} x) = \cos \theta = \sqrt{1-x^2}$.

In the case where $-\frac{\pi}{2} < \theta < 0$, we make the observation that $0 < -\theta < \frac{\pi}{2}$ and hence

Equation:

$$\cos (\sin^{-1} x) = \cos \theta = \cos (-\theta) = \sqrt{1-x^2}.$$

Now if $\theta = \frac{\pi}{2}$ or $\theta = -\frac{\pi}{2}$, $x = 1$ or $x = -1$, and since in either case $\cos \theta = 0$ and $\sqrt{1-x^2} = 0$, we have

Equation:

$$\cos (\sin ^{-1} x)=\cos \theta=\sqrt{1-x^2}.$$

Finally, if $\theta = -$, $x = 0$ and $\cos \theta = \text{mroot} = 1$.

Consequently, in all cases, $\cos (\sin ^{-1} x)=\sqrt{1-x^2}$.

Example:

Exercise:

Problem:

Applying the Chain Rule to the Inverse Sine Function

Apply the chain rule to the formula derived in [\[link\]](#) to find the derivative of $h(x) = \sin^{-1}(g(x))$ and use this result to find the derivative of $h(x) = \sin^{-1}(2x^3)$.

Solution:

Applying the chain rule to $h(x) = \sin^{-1}(g(x))$, we have

Equation:

$$h'(x) = \frac{1}{\sqrt{1-(g(x))^2}} g'(x).$$

Now let $g(x) = 2x^3$, so $g'(x) = 6x^2$. Substituting into the previous result, we obtain

Equation:

$$\begin{aligned} h'(x) &= \frac{1}{\sqrt{1-4x^6}} \cdot 6x^2 \\ &= \frac{6x^2}{\sqrt{1-4x^6}}. \end{aligned}$$

Note:**Exercise:****Problem:**

Use the inverse function theorem to find the derivative of $g(x) = \tan^{-1}x$.

Solution:

$$g'(x) = \frac{1}{1+x^2}$$

Hint

The inverse of $g(x)$ is $f(x) = \tan x$. Use [\[link\]](#) as a guide.

The derivatives of the remaining inverse trigonometric functions may also be found by using the inverse function theorem. These formulas are provided in the following theorem.

Note:**Derivatives of Inverse Trigonometric Functions****Equation:**

$$\frac{d}{dx} \sin^{-1}x = \frac{1}{\sqrt{1-(x)^2}}$$

Equation:

$$\frac{d}{dx} \cos^{-1}x = \frac{-1}{\sqrt{1-(x)^2}}$$

Equation:

$$\frac{d}{dx} \tan^{-1}x = \frac{1}{1+(x)^2}$$

Equation:

$$\frac{d}{dx} \cot^{-1} x = \frac{-1}{1 + (x)^2}$$

Equation:

$$\frac{d}{dx} \sec^{-1} x = \frac{1}{|x| \sqrt{(x)^2 - 1}}$$

Equation:

$$\frac{d}{dx} \csc^{-1} x = \frac{-1}{|x| \sqrt{(x)^2 - 1}}$$

Example:

Exercise:

Problem:

Applying Differentiation Formulas to an Inverse Tangent Function

Find the derivative of $f(x) = \tan^{-1}(x^2)$.

Solution:

Let $g(x) = x^2$, so $g'(x) = 2x$. Substituting into [\[link\]](#), we obtain

Equation:

$$f'(x) = \frac{1}{1 + (x^2)^2} \cdot (2x).$$

Simplifying, we have

Equation:

$$f'(x) = \frac{2x}{1+x^4}.$$

Example:

Exercise:

Problem:

Applying Differentiation Formulas to an Inverse Sine Function

Find the derivative of $h(x) = x^2 \sin^{-1} x$.

Solution:

By applying the product rule, we have

Equation:

$$h'(x) = 2x \sin^{-1} x + \frac{1}{\sqrt{1-x^2}} \cdot x^2.$$

Note:

Exercise:

Problem: Find the derivative of $h(x) = \cos^{-1}(3x - 1)$.

Solution:

$$h'(x) = \frac{-3}{\sqrt{6x-9x^2}}$$

Hint

Use [\[link\]](#). with $g(x) = 3x - 1$

Example:

Exercise:

Problem:

Applying the Inverse Tangent Function

The position of a particle at time t is given by $s(t) = \tan^{-1}\left(\frac{1}{t}\right)$ for $t \geq \frac{1}{2}$. Find the velocity of the particle at time $t = 1$.

Solution:

Begin by differentiating $s(t)$ in order to find $v(t)$. Thus,

Equation:

$$v(t) = s'(t) = \frac{1}{1 + \left(\frac{1}{t}\right)^2} \cdot \frac{-1}{t^2}.$$

Simplifying, we have

Equation:

$$v(t) = -\frac{1}{t^2 + 1}.$$

Thus, $v(1) = -\frac{1}{2}$.

Note:

Exercise:

Problem:

Find the equation of the line tangent to the graph of $f(x) = \sin^{-1}x$ at $x = 0$.

Solution:

$$y = x$$

Hint

$f'(0)$ is the slope of the tangent line.

Key Concepts

- The inverse function theorem allows us to compute derivatives of inverse functions without using the limit definition of the derivative.
- We can use the inverse function theorem to develop differentiation formulas for the inverse trigonometric functions.

Key Equations

- **Inverse function theorem**

$(f^{-1})'(x) = \frac{1}{f'(f^{-1}(x))}$ whenever $f'(f^{-1}(x)) \neq 0$ and $f(x)$ is differentiable.

- **Power rule with rational exponents**

$$\frac{d}{dx} (x^{m/n}) = \frac{m}{n} x^{(m/n)-1}.$$

- **Derivative of inverse sine function**

$$\frac{d}{dx} \sin^{-1} x = \frac{1}{\sqrt{1-x^2}}$$

- **Derivative of inverse cosine function**

$$\frac{d}{dx} \cos^{-1} x = \frac{-1}{\sqrt{1-x^2}}$$

- **Derivative of inverse tangent function**

$$\frac{d}{dx} \tan^{-1} x = \frac{1}{1+x^2}$$

- **Derivative of inverse cotangent function**

$$\frac{d}{dx} \cot^{-1} x = \frac{-1}{1+x^2}$$

- **Derivative of inverse secant function**

$$\frac{d}{dx} \sec^{-1} x = \frac{1}{|x|\sqrt{x^2-1}}$$

- **Derivative of inverse cosecant function**

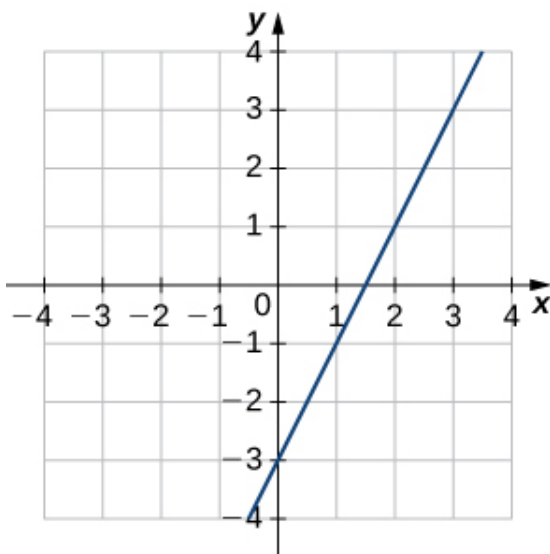
$$\frac{d}{dx} \csc^{-1} x = \frac{-1}{|x|\sqrt{x^2-1}}$$

For the following exercises, use the graph of $y = f(x)$ to

- a. sketch the graph of $y = f^{-1}(x)$, and
b. use part a. to estimate $(f^{-1})'(1)$.

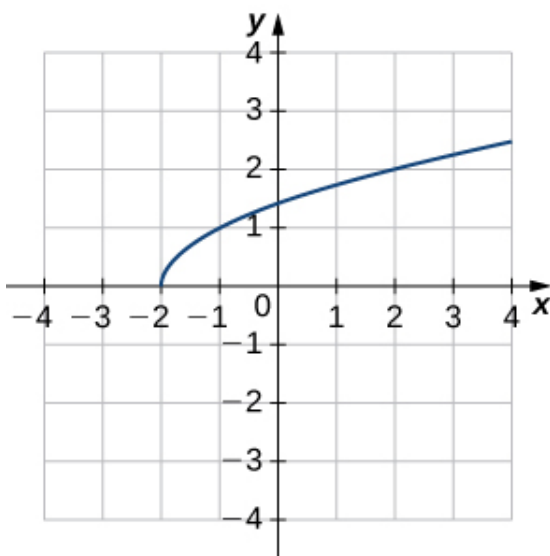
Exercise:

Problem:



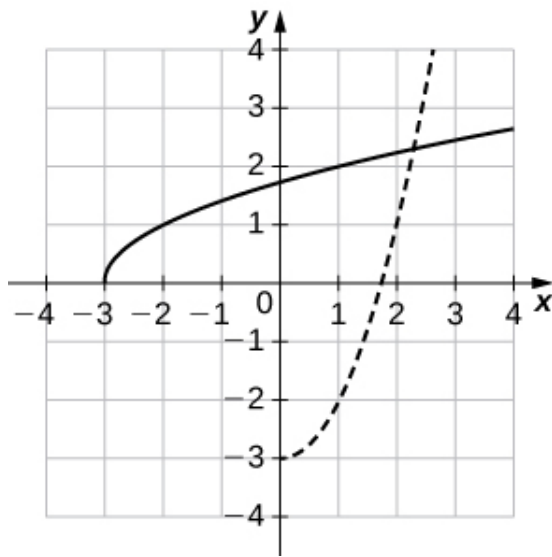
Exercise:

Problem:



Solution:

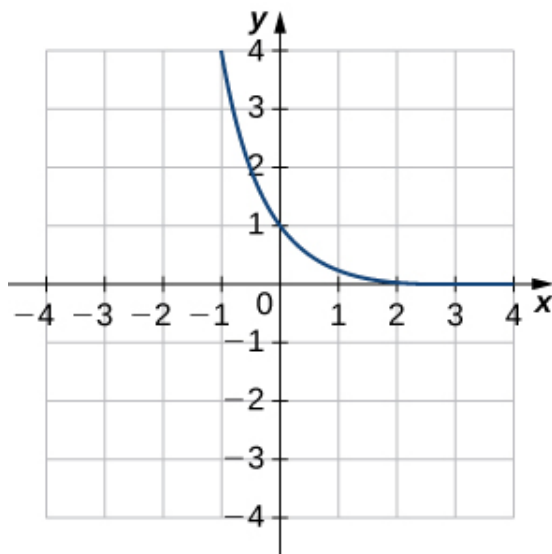
a.



b. $(f^{-1})'(1) \sim 2$

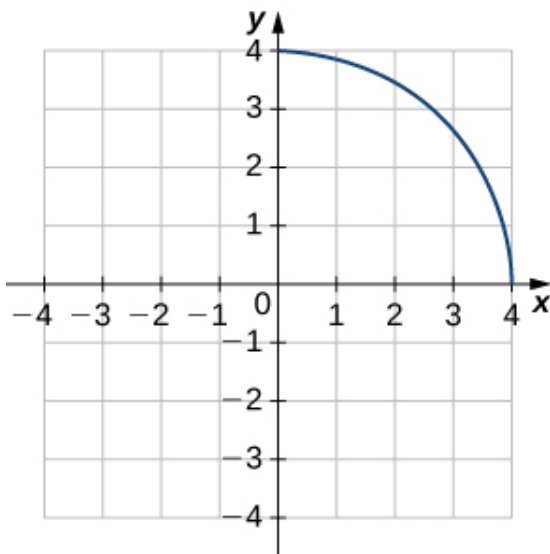
Exercise:

Problem:



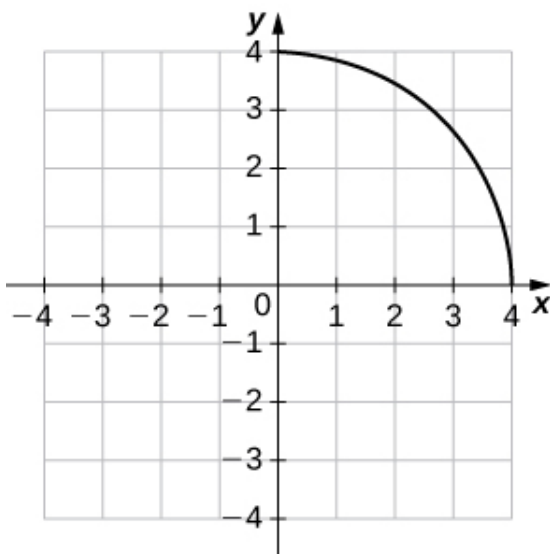
Exercise:

Problem:



Solution:

a.



b. $(f^{-1})'(1) \sim -1/\sqrt{3}$

For the following exercises, use the functions $y = f(x)$ to find

a. $\frac{df}{dx}$ at $x = a$ and

b. $x = f^{-1}(y)$.

c. Then use part b. to find $\frac{df^{-1}}{dy}$ at $y = f(a)$.

Exercise:

Problem: $f(x) = 6x - 1, x = -2$

Exercise:

Problem: $f(x) = 2x^3 - 3, x = 1$

Solution:

a. 6, b. $x = f^{-1}(y) = \left(\frac{y+3}{2}\right)^{1/3}$, c. $\frac{1}{6}$

Exercise:

Problem: $f(x) = 9 - x^2, 0 \leq x \leq 3, x = 2$

Exercise:

Problem: $f(x) = \sin x, x = 0$

Solution:

a. 1, b. $x = f^{-1}(y) = \sin^{-1}y$, c. 1

For each of the following functions, find $(f^{-1})'(a)$.

Exercise:

Problem: $f(x) = x^2 + 3x + 2, x \geq -\frac{3}{2}, a = 2$

Exercise:

Problem: $f(x) = x^3 + 2x + 3, a = 0$

Solution:

$\frac{1}{5}$

Exercise:

Problem: $f(x) = x + \sqrt{x}, a = 2$

Exercise:

Problem: $f(x) = x - \frac{2}{x}, x < 0, a = 1$

Solution:

$$\frac{1}{3}$$

Exercise:

Problem: $f(x) = x + \sin x, a = 0$

Exercise:

Problem: $f(x) = \tan x + 3x^2, a = 0$

Solution:

$$1$$

For each of the given functions $y = f(x)$,

- find the slope of the tangent line to its inverse function f^{-1} at the indicated point P , and
- find the equation of the tangent line to the graph of f^{-1} at the indicated point.

Exercise:

Problem: $f(x) = \frac{4}{1+x^2}, P(2, 1)$

Exercise:

Problem: $f(x) = \sqrt{x-4}, P(2, 8)$

Solution:

a. 4, b. $y = 4x$

Exercise:

Problem: $f(x) = (x^3 + 1)^4, P(16, 1)$

Exercise:

Problem: $f(x) = -x^3 - x + 2, P(-8, 2)$

Solution:

a. $-\frac{1}{96}$, b. $y = -\frac{1}{13}x + \frac{18}{13}$

Exercise:

Problem: $f(x) = x^5 + 3x^3 - 4x - 8, P(-8, 1)$

For the following exercises, find $\frac{dy}{dx}$ for the given function.

Exercise:

Problem: $y = \sin^{-1}(x^2)$

Solution:

$$\frac{2x}{\sqrt{1-x^4}}$$

Exercise:

Problem: $y = \cos^{-1}(\sqrt{x})$

Exercise:

Problem: $y = \sec^{-1}\left(\frac{1}{x}\right)$

Solution:

$$\frac{-1}{\sqrt{1-x^2}}$$

Exercise:

Problem: $y = \sqrt{\csc^{-1} x}$

Exercise:

Problem: $y = (1 + \tan^{-1} x)^3$

Solution:

$$\frac{3(1+\tan^{-1}x)^2}{1+x^2}$$

Exercise:

Problem: $y = \cos^{-1}(2x) \cdot \sin^{-1}(2x)$

Exercise:

Problem: $y = \frac{1}{\tan^{-1}(x)}$

Solution:

$$\frac{-1}{(1+x^2)(\tan^{-1}x)^2}$$

Exercise:

Problem: $y = \sec^{-1}(-x)$

Exercise:

Problem: $y = \cot^{-1}\sqrt{4-x^2}$

Solution:

$$\frac{x}{(5-x^2)\sqrt{4-x^2}}$$

Exercise:

Problem: $y = x \cdot \csc^{-1} x$

For the following exercises, use the given values to find $(f^{-1})'(a)$.

Exercise:

Problem: $f(\pi) = 0, f'(\pi) = -1, a = 0$

Solution:

$$-1$$

Exercise:

Problem: $f(6) = 2, f'(6) = \frac{1}{3}, a = 2$

Exercise:

Problem: $f\left(\frac{1}{3}\right) = -8, f'\left(\frac{1}{3}\right) = 2, a = -8$

Solution:

$$\frac{1}{2}$$

Exercise:

Problem: $f(\sqrt{3}) = \frac{1}{2}, f'(\sqrt{3}) = \frac{2}{3}, a = \frac{1}{2}$

Exercise:

Problem: $f(1) = -3, f'(1) = 10, a = -3$

Solution:

$$\frac{1}{10}$$

Exercise:

Problem: $f(1) = 0, f'(1) = -2, a = 0$

Exercise:**Problem:**

[T] The position of a moving hockey puck after t seconds is $s(t) = \tan^{-1}t$ where s is in meters.

- Find the velocity of the hockey puck at any time t .
- Find the acceleration of the puck at any time t .
- Evaluate a. and b. for $t = 2, 4$, and 6 seconds.
- What conclusion can be drawn from the results in c.?

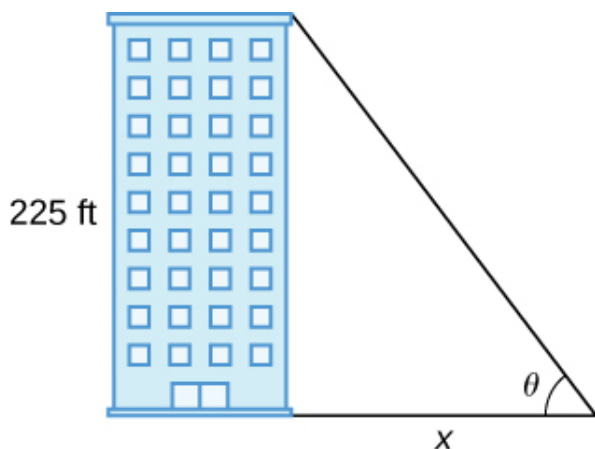
Solution:

a. $v(t) = \frac{1}{1+t^2}$ b. $a(t) = \frac{-2t}{(1+t^2)^2}$ c.

(a) 0.2, 0.06, 0.03; (b) $-0.16, -0.028, -0.0088$ d. The hockey puck is decelerating/slowing down at 2, 4, and 6 seconds.

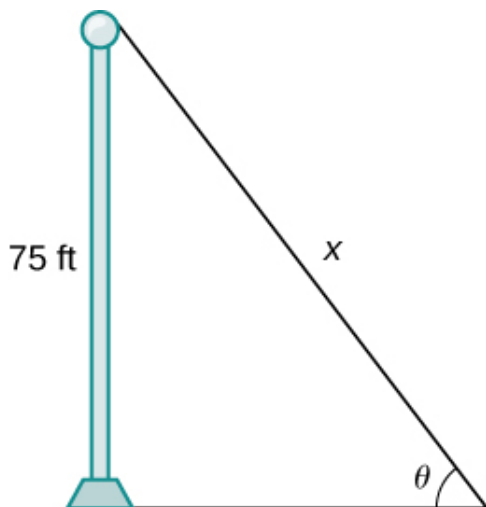
Exercise:**Problem:**

[T] A building that is 225 feet tall casts a shadow of various lengths x as the day goes by. An angle of elevation θ is formed by lines from the top and bottom of the building to the tip of the shadow, as seen in the following figure. Find the rate of change of the angle of elevation $\frac{d\theta}{dx}$ when $x = 272$ feet.



Exercise:**Problem:**

[T] A pole stands 75 feet tall. An angle θ is formed when wires of various lengths of x feet are attached from the ground to the top of the pole, as shown in the following figure. Find the rate of change of the angle $\frac{d\theta}{dx}$ when a wire of length 90 feet is attached.

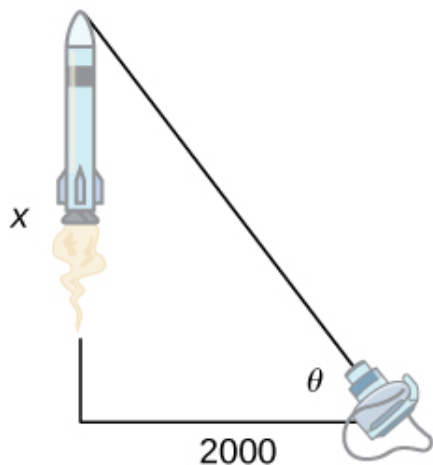


Solution:

−0.0168 radians per foot

Exercise:**Problem:**

[T] A television camera at ground level is 2000 feet away from the launching pad of a space rocket that is set to take off vertically, as seen in the following figure. The angle of elevation of the camera can be found by $\theta = \tan^{-1} \left(\frac{x}{2000} \right)$, where x is the height of the rocket. Find the rate of change of the angle of elevation after launch when the camera and the rocket are 5000 feet apart.

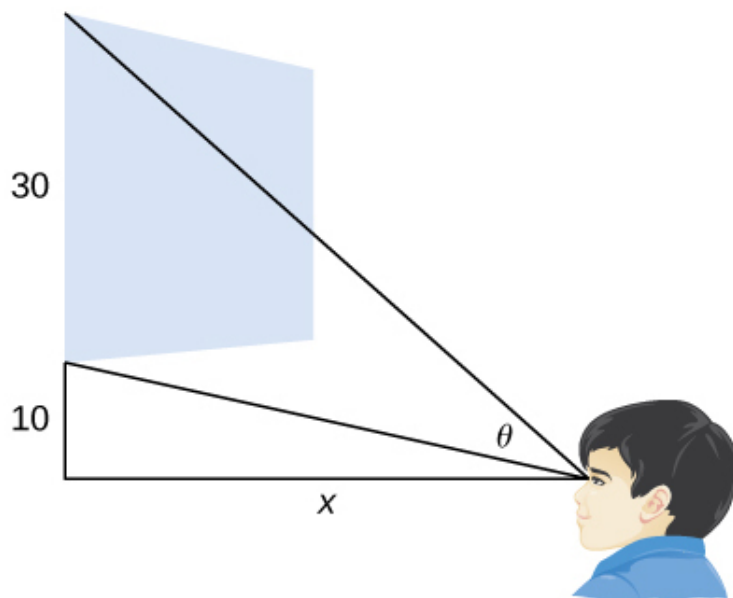


Exercise:

Problem:

[T] A local movie theater with a 30-foot-high screen that is 10 feet above a person's eye level when seated has a viewing angle θ (in radians) given by $\theta = \cot^{-1} \frac{x}{40} - \cot^{-1} \frac{x}{10}$,

where x is the distance in feet away from the movie screen that the person is sitting, as shown in the following figure.



- Find $\frac{d\theta}{dx}$.
- Evaluate $\frac{d\theta}{dx}$ for $x = 5, 10, 15$, and 20 .

- c. Interpret the results in b..
- d. Evaluate $\frac{d\theta}{dx}$ for $x = 25, 30, 35$, and 40
- e. Interpret the results in d. At what distance x should the person stand to maximize his or her viewing angle?

Solution:

- a. $\frac{d\theta}{dx} = \frac{10}{100+x^2} - \frac{40}{1600+x^2}$ b. $\frac{18}{325}, \frac{9}{340}, \frac{42}{4745}, 0$ c. As a person moves farther away from the screen, the viewing angle is increasing, which implies that as he or she moves farther away, his or her screen vision is widening. d. $-\frac{54}{12905}, -\frac{3}{500}, -\frac{198}{29945}, -\frac{9}{1360}$ e. As the person moves beyond 20 feet from the screen, the viewing angle is decreasing. The optimal distance the person should stand for maximizing the viewing angle is 20 feet.

Derivatives of Exponential and Logarithmic Functions

- Find the derivative of exponential functions.
- Find the derivative of logarithmic functions.
- Use logarithmic differentiation to determine the derivative of a function.

So far, we have learned how to differentiate a variety of functions, including trigonometric, inverse, and implicit functions. In this section, we explore derivatives of exponential and logarithmic functions. As we discussed in [Introduction to Functions and Graphs](#), exponential functions play an important role in modeling population growth and the decay of radioactive materials. Logarithmic functions can help rescale large quantities and are particularly helpful for rewriting complicated expressions.

Derivative of the Exponential Function

Just as when we found the derivatives of other functions, we can find the derivatives of exponential and logarithmic functions using formulas. As we develop these formulas, we need to make certain basic assumptions. The proofs that these assumptions hold are beyond the scope of this course.

First of all, we begin with the assumption that the function $B(x) = b^x$, $b > 0$, is defined for every real number and is continuous. In previous courses, the values of exponential functions for all rational numbers were defined—beginning with the definition of b^n , where n is a positive integer—as the product of b multiplied by itself n times. Later, we defined $b^0 = 1$, $b^{-n} = \frac{1}{b^n}$, for a positive integer n , and $b^{s/t} = (\sqrt[t]{b})^s$ for positive integers s and t . These definitions leave open the question of the value of b^r where r is an arbitrary real number. By assuming the continuity of $B(x) = b^x$, $b > 0$, we may interpret b^r as $\lim_{x \rightarrow r} b^x$ where the values of x as we take the limit are rational. For example, we may view 4^π as the number satisfying

Equation:

$$4^3 < 4^\pi < 4^4, 4^{3.1} < 4^\pi < 4^{3.2}, 4^{3.14} < 4^\pi < 4^{3.15}, \\ 4^{3.141} < 4^\pi < 4^{3.142}, 4^{3.1415} < 4^\pi < 4^{3.1416}, \dots$$

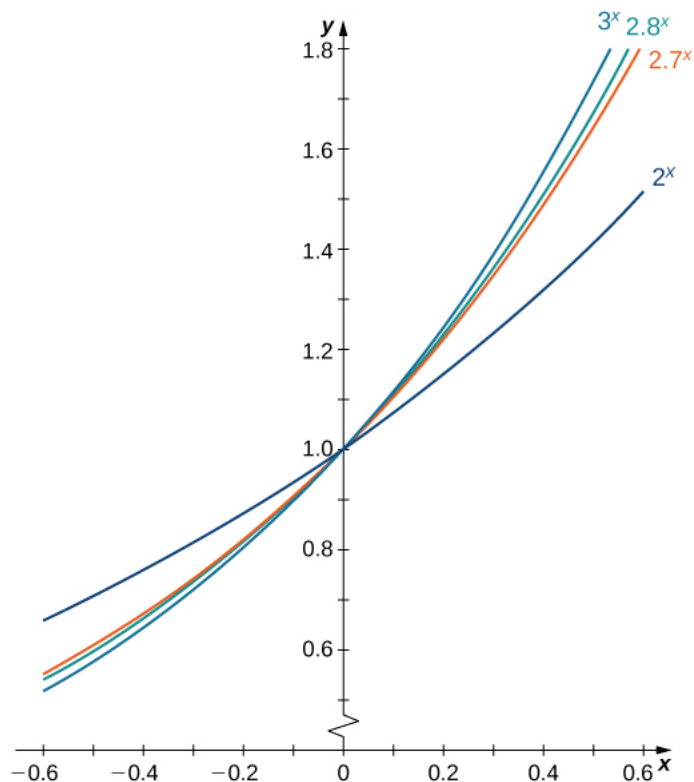
As we see in the following table, $4^\pi \approx 77.88$.

x	4^x	x	4^x
4^3	64	$4^{3.141593}$	77.8802710486
$4^{3.1}$	73.5166947198	$4^{3.1416}$	77.8810268071
$4^{3.14}$	77.7084726013	$4^{3.142}$	77.9242251944
$4^{3.141}$	77.8162741237	$4^{3.15}$	78.7932424541
$4^{3.1415}$	77.8702309526	$4^{3.2}$	84.4485062895
$4^{3.14159}$	77.8799471543	4^4	256

Approximating a Value of 4^π

We also assume that for $B(x) = b^x$, $b > 0$, the value $B'(0)$ of the derivative exists. In this section, we show that by making this one additional assumption, it is possible to prove that the function $B(x)$ is differentiable everywhere.

We make one final assumption: that there is a unique value of $b > 0$ for which $B'(0) = 1$. We define e to be this unique value, as we did in [Introduction to Functions and Graphs](#). [\[link\]](#) provides graphs of the functions $y = 2^x$, $y = 3^x$, $y = 2.7^x$, and $y = 2.8^x$. A visual estimate of the slopes of the tangent lines to these functions at 0 provides evidence that the value of e lies somewhere between 2.7 and 2.8. The function $E(x) = e^x$ is called the **natural exponential function**. Its inverse, $L(x) = \log_e x = \ln x$ is called the **natural logarithmic function**.



The graph of $E(x) = e^x$ is between $y = 2^x$ and $y = 3^x$.

For a better estimate of e , we may construct a table of estimates of $B'(0)$ for functions of the form $B(x) = b^x$. Before doing this, recall that

Equation:

$$B'(0) = \lim_{x \rightarrow 0} \frac{b^x - b^0}{x - 0} = \lim_{x \rightarrow 0} \frac{b^x - 1}{x} \approx \frac{b^x - 1}{x}$$

for values of x very close to zero. For our estimates, we choose $x = 0.00001$ and $x = -0.00001$ to obtain the estimate

Equation:

$$\frac{b^{-0.00001} - 1}{-0.00001} < B'(0) < \frac{b^{0.00001} - 1}{0.00001}.$$

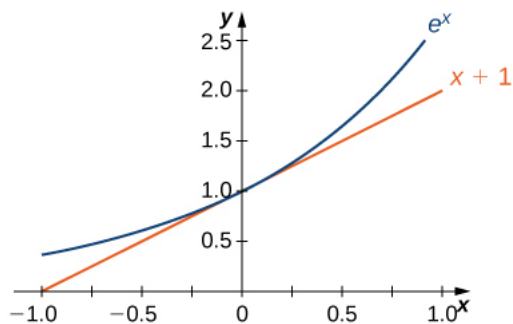
See the following table.

b	$\frac{b^{-0.00001} - 1}{-0.00001} < B'(0) < \frac{b^{0.00001} - 1}{0.00001}$	b	$\frac{b^{-0.00001} - 1}{-0.00001} < B'(0) < \frac{b^{0.00001} - 1}{0.00001}$
2	$0.693145 < B'(0) < 0.69315$	2.7183	$1.000002 < B'(0) < 1.000012$
2.7	$0.993247 < B'(0) < 0.993257$	2.719	$1.000259 < B'(0) < 1.000269$
2.71	$0.996944 < B'(0) < 0.996954$	2.72	$1.000627 < B'(0) < 1.000637$
2.718	$0.999891 < B'(0) < 0.999901$	2.8	$1.029614 < B'(0) < 1.029625$
2.7182	$0.999965 < B'(0) < 0.999975$	3	$1.098606 < B'(0) < 1.098618$

Estimating a Value of e

The evidence from the table suggests that $2.7182 < e < 2.7183$.

The graph of $E(x) = e^x$ together with the line $y = x + 1$ are shown in [\[link\]](#). This line is tangent to the graph of $E(x) = e^x$ at $x = 0$.



The tangent line to $E(x) = e^x$ at $x = 0$ has slope 1.

Now that we have laid out our basic assumptions, we begin our investigation by exploring the derivative of $B(x) = b^x$, $b > 0$. Recall that we have assumed that $B'(0)$ exists. By applying the limit definition to the derivative we conclude that

Equation:

$$B'(0) = \lim_{h \rightarrow 0} \frac{b^{0+h} - b^0}{h} = \lim_{h \rightarrow 0} \frac{b^h - 1}{h}.$$

Turning to $B'(x)$, we obtain the following.

Equation:

$$\begin{aligned}
 B'(x) &= \lim_{h \rightarrow 0} \frac{b^{x+h} - b^x}{h} && \text{Apply the limit definition of the derivative.} \\
 &= \lim_{h \rightarrow 0} \frac{b^x b^h - b^x}{h} && \text{Note that } b^{x+h} = b^x b^h. \\
 &= \lim_{h \rightarrow 0} \frac{b^x (b^h - 1)}{h} && \text{Factor out } b^x. \\
 &= b^x \lim_{h \rightarrow 0} \frac{b^h - 1}{h} && \text{Apply a property of limits.} \\
 &= b^x B'(0) && \text{Use } B'(0) = \lim_{h \rightarrow 0} \frac{b^{0+h} - b^0}{h} = \lim_{h \rightarrow 0} \frac{b^h - 1}{h}.
 \end{aligned}$$

We see that on the basis of the assumption that $B(x) = b^x$ is differentiable at 0, $B(x)$ is not only differentiable everywhere, but its derivative is

Equation:

$$B'(x) = b^x B'(0).$$

For $E(x) = e^x$, $E'(0) = 1$. Thus, we have $E'(x) = e^x$. (The value of $B'(0)$ for an arbitrary function of the form $B(x) = b^x$, $b > 0$, will be derived later.)

Note:

Derivative of the Natural Exponential Function

Let $E(x) = e^x$ be the natural exponential function. Then

Equation:

$$E'(x) = e^x.$$

In general,

Equation:

$$\frac{d}{dx} (e^{g(x)}) = e^{g(x)} g'(x).$$

Example:

Exercise:

Problem:

Derivative of an Exponential Function

Find the derivative of $f(x) = e^{\tan(2x)}$.

Solution:

Using the derivative formula and the chain rule,

Equation:

$$\begin{aligned}
 f'(x) &= e^{\tan(2x)} \frac{d}{dx} (\tan(2x)) \\
 &= e^{\tan(2x)} \sec^2(2x) \cdot 2.
 \end{aligned}$$

Example:**Exercise:****Problem:****Combining Differentiation Rules**

Find the derivative of $y = \frac{e^{x^2}}{x}$.

Solution:

Use the derivative of the natural exponential function, the quotient rule, and the chain rule.

Equation:

$$\begin{aligned} y' &= \frac{(e^{x^2} \cdot 2)x \cdot x - 1 \cdot e^{x^2}}{x^2} && \text{Apply the quotient rule.} \\ &= \frac{e^{x^2}(2x^2 - 1)}{x^2} && \text{Simplify.} \end{aligned}$$

Note:**Exercise:**

Problem: Find the derivative of $h(x) = xe^{2x}$.

Solution:

$$h'(x) = e^{2x} + 2xe^{2x}$$

Hint

Don't forget to use the product rule.

Example:**Exercise:****Problem:****Applying the Natural Exponential Function**

A colony of mosquitoes has an initial population of 1000. After t days, the population is given by $A(t) = 1000e^{0.3t}$. Show that the ratio of the rate of change of the population, $A'(t)$, to the population, $A(t)$ is constant.

Solution:

First find $A'(t)$. By using the chain rule, we have $A'(t) = 300e^{0.3t}$. Thus, the ratio of the rate of change of the population to the population is given by

Equation:

$$A'(t) = \frac{300e^{0.3t}}{1000e^{0.3t}} = 0.3.$$

The ratio of the rate of change of the population to the population is the constant 0.3.

Note:

Exercise:

Problem:

If $A(t) = 1000e^{0.3t}$ describes the mosquito population after t days, as in the preceding example, what is the rate of change of $A(t)$ after 4 days?

Solution:

996

Hint

Find $A'(4)$.

Derivative of the Logarithmic Function

Now that we have the derivative of the natural exponential function, we can use implicit differentiation to find the derivative of its inverse, the natural logarithmic function.

Note:

The Derivative of the Natural Logarithmic Function

If $x > 0$ and $y = \ln x$, then

Equation:

$$\frac{dy}{dx} = \frac{1}{x}.$$

More generally, let $g(x)$ be a differentiable function. For all values of x for which $g'(x) > 0$, the derivative of $h(x) = \ln(g(x))$ is given by

Equation:

$$h'(x) = \frac{1}{g(x)} g'(x).$$

Proof

If $x > 0$ and $y = \ln x$, then $e^y = x$. Differentiating both sides of this equation results in the equation

Equation:

$$e^y \frac{dy}{dx} = 1.$$

Solving for $\frac{dy}{dx}$ yields

Equation:

$$\frac{dy}{dx} = \frac{1}{e^y}.$$

Finally, we substitute $x = e^y$ to obtain

Equation:

$$\frac{dy}{dx} = \frac{1}{x}.$$

We may also derive this result by applying the inverse function theorem, as follows. Since $y = g(x) = \ln x$ is the inverse of $f(x) = e^x$, by applying the inverse function theorem we have

Equation:

$$\frac{dy}{dx} = \frac{1}{f'(g(x))} = \frac{1}{e^{\ln x}} = \frac{1}{x}.$$

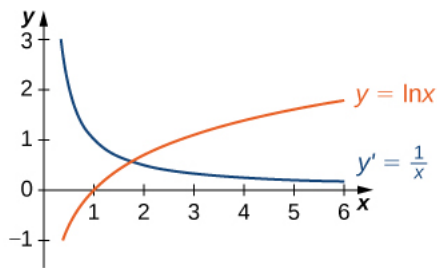
Using this result and applying the chain rule to $h(x) = \ln(g(x))$ yields

Equation:

$$h'(x) = \frac{1}{g(x)} g'(x).$$

□

The graph of $y = \ln x$ and its derivative $\frac{dy}{dx} = \frac{1}{x}$ are shown in [\[link\]](#).



The function $y = \ln x$ is increasing on $(0, +\infty)$. Its derivative $y' = \frac{1}{x}$ is greater than zero on $(0, +\infty)$.

Example:

Exercise:

Problem:

Taking a Derivative of a Natural Logarithm

Find the derivative of $f(x) = \ln(x^3 + 3x - 4)$.

Solution:

Use [link](#) directly.

Equation:

$$\begin{aligned} f'(x) &= \frac{1}{x^3+3x-4} \cdot (3x^2 + 3) && \text{Use } g(x) = x^3 + 3x - 4 \text{ in } h'(x) = \frac{1}{g(x)} g'(x). \\ &= \frac{3x^2+3}{x^3+3x-4} && \text{Rewrite.} \end{aligned}$$

Example:

Exercise:

Problem:

Using Properties of Logarithms in a Derivative

Find the derivative of $f(x) = \ln\left(\frac{x^2 \sin x}{2x+1}\right)$.

Solution:

At first glance, taking this derivative appears rather complicated. However, by using the properties of logarithms prior to finding the derivative, we can make the problem much simpler.

Equation:

$$\begin{aligned} f(x) &= \ln\left(\frac{x^2 \sin x}{2x+1}\right) = 2 \ln x + \ln(\sin x) - \ln(2x+1) && \text{Apply properties of logarithms.} \\ f'(x) &= \frac{2}{x} + \cot x - \frac{2}{2x+1} && \text{Apply sum rule and } h'(x) = \frac{1}{g(x)} g'(x). \end{aligned}$$

Note:

Exercise:

Problem: Differentiate: $f(x) = \ln(3x + 2)^5$.

Solution:

$$f'(x) = \frac{15}{3x+2}$$

Hint

Use a property of logarithms to simplify before taking the derivative.

Now that we can differentiate the natural logarithmic function, we can use this result to find the derivatives of $y = \log_b x$ and $y = b^x$ for $b > 0, b \neq 1$.

Note:**Derivatives of General Exponential and Logarithmic Functions**

Let $b > 0$, $b \neq 1$, and let $g(x)$ be a differentiable function.

i. If $y = \log_b x$, then

Equation:

$$\frac{dy}{dx} = \frac{1}{x \ln b}.$$

More generally, if $h(x) = \log_b(g(x))$, then for all values of x for which $g(x) > 0$,

Equation:

$$h'(x) = \frac{g'(x)}{g(x) \ln b}.$$

ii. If $y = b^x$, then

Equation:

$$\frac{dy}{dx} = b^x \ln b.$$

More generally, if $h(x) = b^{g(x)}$, then

Equation:

$$h'(x) = b^{g(x)} g'(x) \ln b.$$

Proof

If $y = \log_b x$, then $b^y = x$. It follows that $\ln(b^y) = \ln x$. Thus $y \ln b = \ln x$. Solving for y , we have $y = \frac{\ln x}{\ln b}$.

Differentiating and keeping in mind that $\ln b$ is a constant, we see that

Equation:

$$\frac{dy}{dx} = \frac{1}{x \ln b}.$$

The derivative in [\[link\]](#) now follows from the chain rule.

If $y = b^x$, then $\ln y = x \ln b$. Using implicit differentiation, again keeping in mind that $\ln b$ is constant, it follows that $\frac{1}{y} \frac{dy}{dx} = \ln b$. Solving for $\frac{dy}{dx}$ and substituting $y = b^x$, we see that

Equation:

$$\frac{dy}{dx} = y \ln b = b^x \ln b.$$

The more general derivative ([\[link\]](#)) follows from the chain rule.

□

Example:

Exercise:

Problem:

Applying Derivative Formulas

Find the derivative of $h(x) = \frac{3^x}{3^x + 2}$.

Solution:

Use the quotient rule and [\[link\]](#).

Equation:

$$\begin{aligned} h'(x) &= \frac{3^x \ln 3(3^x + 2) - 3^x \ln 3(3^x)}{(3^x + 2)^2} && \text{Apply the quotient rule.} \\ &= \frac{2 \cdot 3^x \ln 3}{(3^x + 2)^2} && \text{Simplify.} \end{aligned}$$

Example:

Exercise:

Problem:

Finding the Slope of a Tangent Line

Find the slope of the line tangent to the graph of $y = \log_2(3x + 1)$ at $x = 1$.

Solution:

To find the slope, we must evaluate $\frac{dy}{dx}$ at $x = 1$. Using [\[link\]](#), we see that

Equation:

$$\frac{dy}{dx} = \frac{3}{(3x + 1) \ln 2}.$$

By evaluating the derivative at $x = 1$, we see that the tangent line has slope

Equation:

$$\left. \frac{dy}{dx} \right|_{x=1} = \frac{3}{4 \ln 2} = \frac{3}{\ln 16}.$$

Note:

Exercise:

Problem: Find the slope for the line tangent to $y = 3^x$ at $x = 2$.

Solution:

$$9 \ln(3)$$

Hint

Evaluate the derivative at $x = 2$.

Logarithmic Differentiation

At this point, we can take derivatives of functions of the form $y = (g(x))^n$ for certain values of n , as well as functions of the form $y = b^{g(x)}$, where $b > 0$ and $b \neq 1$. Unfortunately, we still do not know the derivatives of functions such as $y = x^x$ or $y = x^\pi$. These functions require a technique called **logarithmic differentiation**, which allows us to differentiate any function of the form $h(x) = g(x)^{f(x)}$. It can also be used to convert a very complex differentiation problem into a simpler one, such as finding the derivative of $y = \frac{x\sqrt{2x+1}}{e^x \sin^3 x}$. We outline this technique in the following problem-solving strategy.

Note:

Problem-Solving Strategy: Using Logarithmic Differentiation

1. To differentiate $y = h(x)$ using logarithmic differentiation, take the natural logarithm of both sides of the equation to obtain $\ln y = \ln(h(x))$.
2. Use properties of logarithms to expand $\ln(h(x))$ as much as possible.
3. Differentiate both sides of the equation. On the left we will have $\frac{1}{y} \frac{dy}{dx}$.
4. Multiply both sides of the equation by y to solve for $\frac{dy}{dx}$.
5. Replace y by $h(x)$.

Example:

Exercise:

Problem:

Using Logarithmic Differentiation

Find the derivative of $y = (2x^4 + 1)^{\tan x}$.

Solution:

Use logarithmic differentiation to find this derivative.

Equation:

$$\ln y = \ln(2x^4 + 1)^{\tan x}$$

$$\ln y = \tan x \ln(2x^4 + 1)$$

$$\frac{1}{y} \frac{dy}{dx} = \sec^2 x \ln(2x^4 + 1) + \frac{8x^3}{2x^4 + 1} \cdot \tan x$$

$$\frac{dy}{dx} = y \cdot \left(\sec^2 x \ln(2x^4 + 1) + \frac{8x^3}{2x^4 + 1} \cdot \tan x \right)$$

$$\frac{dy}{dx} = (2x^4 + 1)^{\tan x} \left(\sec^2 x \ln(2x^4 + 1) + \frac{8x^3}{2x^4 + 1} \cdot \tan x \right)$$

Step 1. Take the natural logarithm of both sides.

Step 2. Expand using properties of logarithms.

Step 3. Differentiate both sides. Use the product rule on the right.

Step 4. Multiply by y on both sides.

Step 5. Substitute $y = (2x^4 + 1)^{\tan x}$.

Example:

Exercise:**Problem:**
Using Logarithmic Differentiation

Find the derivative of $y = \frac{x\sqrt{2x+1}}{e^x \sin^3 x}$.

Solution:

This problem really makes use of the properties of logarithms and the differentiation rules given in this chapter.

Equation:

$\ln y = \ln \frac{x\sqrt{2x+1}}{e^x \sin^3 x}$	Step 1. Take the natural logarithm of both sides.
$\ln y = \ln x + \frac{1}{2} \ln(2x+1) - x \ln e - 3 \ln \sin x$	Step 2. Expand using properties of logarithms.
$\frac{1}{y} \frac{dy}{dx} = \frac{1}{x} + \frac{1}{2x+1} - 1 - 3 \frac{\cos x}{\sin x}$	Step 3. Differentiate both sides.
$\frac{dy}{dx} = y \left(\frac{1}{x} + \frac{1}{2x+1} - 1 - 3 \cot x \right)$	Step 4. Multiply by y on both sides.
$\frac{dy}{dx} = \frac{x\sqrt{2x+1}}{e^x \sin^3 x} \left(\frac{1}{x} + \frac{1}{2x+1} - 1 - 3 \cot x \right)$	Step 5. Substitute $y = \frac{x\sqrt{2x+1}}{e^x \sin^3 x}$.

Example:**Exercise:****Problem:**
Extending the Power Rule

Find the derivative of $y = x^r$ where r is an arbitrary real number.

Solution:

The process is the same as in [\[link\]](#), though with fewer complications.

Equation:

$\ln y = \ln x^r$	Step 1. Take the natural logarithm of both sides.
$\ln y = r \ln x$	Step 2. Expand using properties of logarithms.
$\frac{1}{y} \frac{dy}{dx} = r \frac{1}{x}$	Step 3. Differentiate both sides.
$\frac{dy}{dx} = y \frac{r}{x}$	Step 4. Multiply by y on both sides.
$\frac{dy}{dx} = x^r \frac{r}{x}$	Step 5. Substitute $y = x^r$.
$\frac{dy}{dx} = r x^{r-1}$	Simplify.

Note:**Exercise:**

Problem: Use logarithmic differentiation to find the derivative of $y = x^x$.

Solution:

$$\frac{dy}{dx} = x^x (1 + \ln x)$$

Hint

Follow the problem solving strategy.

Note:**Exercise:**

Problem: Find the derivative of $y = (\tan x)^\pi$.

Solution:

$$y' = \pi(\tan x)^{\pi-1} \sec^2 x$$

Hint

Use the result from [\[link\]](#).

Key Concepts

- On the basis of the assumption that the exponential function $y = b^x, b > 0$ is continuous everywhere and differentiable at 0, this function is differentiable everywhere and there is a formula for its derivative.
- We can use a formula to find the derivative of $y = \ln x$, and the relationship $\log_b x = \frac{\ln x}{\ln b}$ allows us to extend our differentiation formulas to include logarithms with arbitrary bases.
- Logarithmic differentiation allows us to differentiate functions of the form $y = g(x)^{f(x)}$ or very complex functions by taking the natural logarithm of both sides and exploiting the properties of logarithms before differentiating.

Key Equations

- **Derivative of the natural exponential function**
$$\frac{d}{dx} (e^{g(x)}) = e^{g(x)} g'(x)$$
- **Derivative of the natural logarithmic function**
$$\frac{d}{dx} (\ln g(x)) = \frac{1}{g(x)} g'(x)$$
- **Derivative of the general exponential function**
$$\frac{d}{dx} (b^{g(x)}) = b^{g(x)} g'(x) \ln b$$
- **Derivative of the general logarithmic function**
$$\frac{d}{dx} (\log_b g(x)) = \frac{g'(x)}{g(x) \ln b}$$

For the following exercises, find $f'(x)$ for each function.

Exercise:

Problem: $f(x) = x^2 e^x$

Solution:

$$2xe^x + x^2 e^x$$

Exercise:

Problem: $f(x) = \frac{e^{-x}}{x}$

Exercise:

Problem: $f(x) = e^{x^3 \ln x}$

Solution:

$$e^{x^3 \ln x} (3x^2 \ln x + x^2)$$

Exercise:

Problem: $f(x) = \sqrt{e^{2x} + 2x}$

Exercise:

Problem: $f(x) = \frac{e^x - e^{-x}}{e^x + e^{-x}}$

Solution:

$$\frac{4}{(e^x + e^{-x})^2}$$

Exercise:

Problem: $f(x) = \frac{10^x}{\ln 10}$

Exercise:

Problem: $f(x) = 2^{4x} + 4x^2$

Solution:

$$2^{4x+2} \cdot \ln 2 + 8x$$

Exercise:

Problem: $f(x) = 3^{\sin 3x}$

Exercise:

Problem: $f(x) = x^\pi \cdot \pi^x$

Solution:

$$\pi x^{\pi-1} \cdot \pi^x + x^\pi \cdot \pi^x \ln \pi$$

Exercise:

Problem: $f(x) = \ln(4x^3 + x)$

Exercise:

Problem: $f(x) = \ln \sqrt{5x - 7}$

Solution:

$$\frac{5}{2(5x-7)}$$

Exercise:

Problem: $f(x) = x^2 \ln 9x$

Exercise:

Problem: $f(x) = \log(\sec x)$

Solution:

$$\frac{\tan x}{\ln 10}$$

Exercise:

Problem: $f(x) = \log_7(6x^4 + 3)^5$

Exercise:

Problem: $f(x) = 2^x \cdot \log_3 7^{x^2-4}$

Solution:

$$2^x \cdot \ln 2 \cdot \log_3 7^{x^2-4} + 2^x \cdot \frac{2x \ln 7}{\ln 3}$$

For the following exercises, use logarithmic differentiation to find $\frac{dy}{dx}$.

Exercise:

Problem: $y = x^{\sqrt{x}}$

Exercise:

Problem: $y = (\sin 2x)^{4x}$

Solution:

$$(\sin 2x)^{4x} [4 \cdot \ln(\sin 2x) + 8x \cdot \cot 2x]$$

Exercise:

Problem: $y = (\ln x)^{\ln x}$

Exercise:

Problem: $y = x^{\log_2 x}$

Solution:

$$x^{\log_2 x} \cdot \frac{2 \ln x}{x \ln 2}$$

Exercise:

Problem: $y = (x^2 - 1)^{\ln x}$

Exercise:

Problem: $y = x^{\cot x}$

Solution:

$$x^{\cot x} \cdot \left[-\csc^2 x \cdot \ln x + \frac{\cot x}{x} \right]$$

Exercise:

Problem: $y = \frac{x+11}{\sqrt[3]{x^2-4}}$

Exercise:

Problem: $y = x^{-1/2}(x^2 + 3)^{2/3}(3x - 4)^4$

Solution:

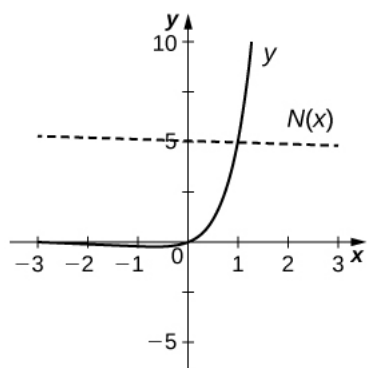
$$x^{-1/2}(x^2 + 3)^{2/3}(3x - 4)^4 \cdot \left[\frac{-1}{2x} + \frac{4x}{3(x^2+3)} + \frac{12}{3x-4} \right]$$

Exercise:

Problem: [T] Find an equation of the tangent line to the graph of $f(x) = 4xe^{(x^2-1)}$ at the point where $x = -1$. Graph both the function and the tangent line.

Exercise:**Problem:**

[T] Find the equation of the line that is normal to the graph of $f(x) = x \cdot 5^x$ at the point where $x = 1$. Graph both the function and the normal line.

Solution:

$$y = \frac{-1}{5+5\ln 5}x + \left(5 + \frac{1}{5+5\ln 5}\right)$$

Exercise:

Problem:

[T] Find the equation of the tangent line to the graph of $x^3 - x \ln y + y^3 = 2x + 5$ at the point where $x = 2$.
 (Hint: Use implicit differentiation to find $\frac{dy}{dx}$.) Graph both the curve and the tangent line.

Exercise:

Problem: Consider the function $y = x^{1/x}$ for $x > 0$.

- Determine the points on the graph where the tangent line is horizontal.
- Determine the points on the graph where $y' > 0$ and those where $y' < 0$.

Solution:

- $x = e^{-2.718}$
- $(e, \infty), (0, e)$

Exercise:

Problem: The formula $I(t) = \frac{\sin t}{e^t}$ is the formula for a decaying alternating current.

- Complete the following table with the appropriate values.

t	$\frac{\sin t}{e^t}$
0	(i)
$\frac{\pi}{2}$	(ii)
π	(iii)
$\frac{3\pi}{2}$	(iv)
2π	(v)
$\frac{5\pi}{2}$	(vi)
3π	(vii)
$\frac{7\pi}{2}$	(viii)
4π	(ix)

- Using only the values in the table, determine where the tangent line to the graph of $I(t)$ is horizontal.

Exercise:

Problem:

[T] The population of Toledo, Ohio, in 2000 was approximately 500,000. Assume the population is increasing at a rate of 5% per year.

- Write the exponential function that relates the total population as a function of t .
- Use a. to determine the rate at which the population is increasing in t years.
- Use b. to determine the rate at which the population is increasing in 10 years.

Solution:

a. $P = 500,000(1.05)^t$ individuals b. $P'(t) = 24395 \cdot (1.05)^t$ individuals per year c. 39,737 individuals per year

Exercise:**Problem:**

[T] An isotope of the element erbium has a half-life of approximately 12 hours. Initially there are 9 grams of the isotope present.

- Write the exponential function that relates the amount of substance remaining as a function of t , measured in hours.
- Use a. to determine the rate at which the substance is decaying in t hours.
- Use b. to determine the rate of decay at $t = 4$ hours.

Exercise:**Problem:**

[T] The number of cases of influenza in New York City from the beginning of 1960 to the beginning of 1961 is modeled by the function

$N(t) = 5.3e^{0.093t^2 - 0.87t}$, ($0 \leq t \leq 4$), where $N(t)$ gives the number of cases (in thousands) and t is measured in years, with $t = 0$ corresponding to the beginning of 1960.

- Show work that evaluates $N(0)$ and $N(4)$. Briefly describe what these values indicate about the disease in New York City.
- Show work that evaluates $N'(0)$ and $N'(3)$. Briefly describe what these values indicate about the disease in New York City.

Solution:

a. At the beginning of 1960 there were 5.3 thousand cases of the disease in New York City. At the beginning of 1964 there were approximately 723 cases of the disease in the United States. b. At the beginning of 1960 the number of cases of the disease was decreasing at rate of -4.611 thousand per year; at the beginning of 1963, the number of cases of the disease was decreasing at a rate of -0.2808 thousand per year.

Exercise:**Problem:**

[T] The *relative rate of change* of a differentiable function $y = f(x)$ is given by $\frac{100 \cdot f'(x)}{f(x)}\%$. One model for population growth is a Gompertz growth function, given by $P(x) = ae^{-b \cdot e^{-cx}}$ where a , b , and c are constants.

- Find the relative rate of change formula for the generic Gompertz function.

- b. Use a. to find the relative rate of change of a population in $x = 20$ months when $a = 204$, $b = 0.0198$, and $c = 0.15$.
- c. Briefly interpret what the result of b. means.

For the following exercises, use the population of New York City from 1790 to 1860, given in the following table.

Years since 1790	Population
0	33,131
10	60,515
20	96,373
30	123,706
40	202,300
50	312,710
60	515,547
70	813,669

New York City Population Over TimeSource: http://en.wikipedia.org/wiki/Largest_cities_in_the_United_States_by_population_by_decade.

Exercise:

Problem: [T] Using a computer program or a calculator, fit a growth curve to the data of the form $p = ab^t$.

Solution:

$$p = 35741(1.045)^t$$

Exercise:

Problem:

[T] Using the exponential best fit for the data, write a table containing the derivatives evaluated at each year.

Exercise:

Problem:

[T] Using the exponential best fit for the data, write a table containing the second derivatives evaluated at each year.

Solution:

Years since 1790	P''
0	69.25
10	107.5
20	167.0
30	259.4
40	402.8
50	625.5
60	971.4
70	1508.5

Exercise:

Problem: [T] Using the tables of first and second derivatives and the best fit, answer the following questions:

- Will the model be accurate in predicting the future population of New York City? Why or why not?
- Estimate the population in 2010. Was the prediction correct from a.?

Chapter Review Exercises

True or False? Justify the answer with a proof or a counterexample.

Exercise:

Problem: Every function has a derivative.

Solution:

False.

Exercise:

Problem: A continuous function has a continuous derivative.

Exercise:

Problem: A continuous function has a derivative.

Solution:

False

Exercise:

Problem: If a function is differentiable, it is continuous.

Use the limit definition of the derivative to exactly evaluate the derivative.

Exercise:

Problem: $f(x) = \sqrt{x+4}$

Solution:

$$\frac{1}{2\sqrt{x+4}}$$

Exercise:

Problem: $f(x) = \frac{3}{x}$

Find the derivatives of the following functions.

Exercise:

Problem: $f(x) = 3x^3 - \frac{4}{x^2}$

Solution:

$$9x^2 + \frac{8}{x^3}$$

Exercise:

Problem: $f(x) = (4 - x^2)^3$

Exercise:

Problem: $f(x) = e^{\sin x}$

Solution:

$$e^{\sin x} \cos x$$

Exercise:

Problem: $f(x) = \ln(x+2)$

Exercise:

Problem: $f(x) = x^2 \cos x + x \tan(x)$

Solution:

$$x \sec^2(x) + 2x \cos(x) + \tan(x) - x^2 \sin(x)$$

Exercise:

Problem: $f(x) = \sqrt{3x^2 + 2}$

Exercise:

Problem: $f(x) = \frac{x}{4} \sin^{-1}(x)$

Solution:

$$\frac{1}{4} \left(\frac{x}{\sqrt{1-x^2}} + \sin^{-1}(x) \right)$$

Exercise:

Problem: $x^2y = (y + 2) + xy \sin(x)$

Find the following derivatives of various orders.

Exercise:

Problem: First derivative of $y = x \ln(x) \cos x$

Solution:

$$\cos x \cdot (\ln x + 1) - x \ln(x) \sin x$$

Exercise:

Problem: Third derivative of $y = (3x + 2)^2$

Exercise:

Problem: Second derivative of $y = 4^x + x^2 \sin(x)$

Solution:

$$4^x (\ln 4)^2 + 2 \sin x + 4x \cos x - x^2 \sin x$$

Find the equation of the tangent line to the following equations at the specified point.

Exercise:

Problem: $y = \cos^{-1}(x) + x$ at $x = 0$

Exercise:

Problem: $y = x + e^x - \frac{1}{x}$ at $x = 1$

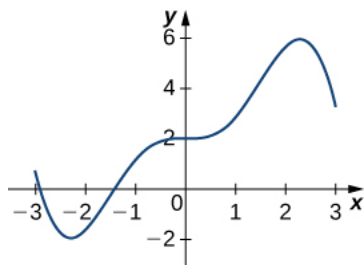
Solution:

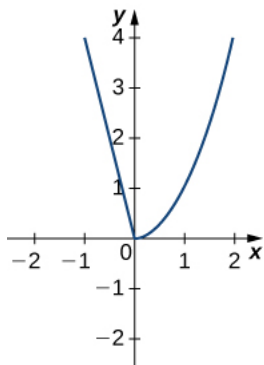
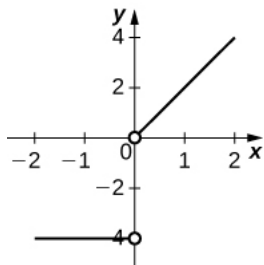
$$T = (2 + e)x - 2$$

Draw the derivative for the following graphs.

Exercise:

Problem:



Exercise:**Problem:****Solution:**

The following questions concern the water level in Ocean City, New Jersey, in January, which can be approximated by $w(t) = 1.9 + 2.9 \cos\left(\frac{\pi}{6}t\right)$, where t is measured in hours after midnight, and the height is measured in feet.

Exercise:

Problem: Find and graph the derivative. What is the physical meaning?

Exercise:

Problem: Find $w'(3)$. What is the physical meaning of this value?

Solution:

$w'(3) = -\frac{2.9\pi}{6}$. At 3 a.m. the tide is decreasing at a rate of 1.514 ft/hr.

The following questions consider the wind speeds of Hurricane Katrina, which affected New Orleans, Louisiana, in August 2005. The data are displayed in a table.

Hours after Midnight, August 26	Wind Speed (mph)
1	45
5	75
11	100
29	115
49	145
58	175
73	155
81	125
85	95
107	35

Wind Speeds of Hurricane KatrinaSource:

http://news.nationalgeographic.com/news/2005/09/0914_050914_katrina_timeline.html.

Exercise:

Problem:

Using the table, estimate the derivative of the wind speed at hour 39. What is the physical meaning?

Exercise:

Problem: Estimate the derivative of the wind speed at hour 83. What is the physical meaning?

Solution:

−7.5. The wind speed is decreasing at a rate of 7.5 mph/hr

Glossary

logarithmic differentiation

is a technique that allows us to differentiate a function by first taking the natural logarithm of both sides of an equation, applying properties of logarithms to simplify the equation, and differentiating implicitly

Antiderivatives

- Find the general antiderivative of a given function.
- Explain the terms and notation used for an indefinite integral.
- State the power rule for integrals.
- Use antidifferentiation to solve simple initial-value problems.

At this point, we have seen how to calculate derivatives of many functions and have been introduced to a variety of their applications. We now ask a question that turns this process around: Given a function f , how do we find a function with the derivative f and why would we be interested in such a function?

We answer the first part of this question by defining antiderivatives. The antiderivative of a function f is a function with a derivative f . Why are we interested in antiderivatives? The need for antiderivatives arises in many situations, and we look at various examples throughout the remainder of the text. Here we examine one specific example that involves rectilinear motion. In our examination in [Derivatives](#) of rectilinear motion, we showed that given a position function $s(t)$ of an object, then its velocity function $v(t)$ is the derivative of $s(t)$ —that is, $v(t) = s'(t)$. Furthermore, the acceleration $a(t)$ is the derivative of the velocity $v(t)$ —that is, $a(t) = v'(t) = s''(t)$. Now suppose we are given an acceleration function a , but not the velocity function v or the position function s . Since $a(t) = v'(t)$, determining the velocity function requires us to find an antiderivative of the acceleration function. Then, since $v(t) = s'(t)$, determining the position function requires us to find an antiderivative of the velocity function. Rectilinear motion is just one case in which the need for antiderivatives arises. We will see many more examples throughout the remainder of the text. For now, let's look at the terminology and notation for antiderivatives, and determine the antiderivatives for several types of functions. We examine various techniques for finding antiderivatives of more complicated functions later in the text ([Introduction to Techniques of Integration](#)).

The Reverse of Differentiation

At this point, we know how to find derivatives of various functions. We now ask the opposite question. Given a function f , how can we find a function with derivative f ? If we can find a function F with derivative f , we call F an antiderivative of f .

Note:

Definition

A function F is an **antiderivative** of the function f if

Equation:

$$F'(x) = f(x)$$

for all x in the domain of f .

Consider the function $f(x) = 2x$. Knowing the power rule of differentiation, we conclude that $F(x) = x^2$ is an antiderivative of f since $F'(x) = 2x$. Are there any other antiderivatives of f ? Yes; since the derivative of any constant C is zero, $x^2 + C$ is also an antiderivative of $2x$. Therefore,

$x^2 + 5$ and $x^2 - \sqrt{2}$ are also antiderivatives. Are there any others that are not of the form $x^2 + C$ for some constant C ? The answer is no. From Corollary 2 of the Mean Value Theorem, we know that if F and G are differentiable functions such that $F'(x) = G'(x)$, then $F(x) - G(x) = C$ for some constant C . This fact leads to the following important theorem.

Note:

General Form of an Antiderivative

Let F be an antiderivative of f over an interval I . Then,

- i. for each constant C , the function $F(x) + C$ is also an antiderivative of f over I ;
- ii. if G is an antiderivative of f over I , there is a constant C for which $G(x) = F(x) + C$ over I .

In other words, the most general form of the antiderivative of f over I is $F(x) + C$.

We use this fact and our knowledge of derivatives to find all the antiderivatives for several functions.

Example:

Exercise:

Problem:

Finding Antiderivatives

For each of the following functions, find all antiderivatives.

- a. $f(x) = 3x^2$
- b. $f(x) = \frac{1}{x}$
- c. $f(x) = \cos x$
- d. $f(x) = e^x$

Solution:

- a. Because

Equation:

$$\frac{d}{dx}(x^3) = 3x^2$$

then $F(x) = x^3$ is an antiderivative of $3x^2$. Therefore, every antiderivative of $3x^2$ is of the form $x^3 + C$ for some constant C , and every function of the form $x^3 + C$ is an antiderivative of $3x^2$.

- b. Let $f(x) = \ln|x|$. For $x > 0$, $f(x) = \ln(x)$ and

Equation:

$$\frac{d}{dx}(\ln x) = \frac{1}{x}.$$

For $x < 0$, $f(x) = \ln(-x)$ and

Equation:

$$\frac{d}{dx}(\ln(-x)) = -\frac{1}{-x} = \frac{1}{x}.$$

Therefore,

Equation:

$$\frac{d}{dx}(\ln|x|) = \frac{1}{x}.$$

Thus, $F(x) = \ln|x|$ is an antiderivative of $\frac{1}{x}$. Therefore, every antiderivative of $\frac{1}{x}$ is of the form $\ln|x| + C$ for some constant C and every function of the form $\ln|x| + C$ is an antiderivative of $\frac{1}{x}$.

c. We have

Equation:

$$\frac{d}{dx}(\sin x) = \cos x,$$

so $F(x) = \sin x$ is an antiderivative of $\cos x$. Therefore, every antiderivative of $\cos x$ is of the form $\sin x + C$ for some constant C and every function of the form $\sin x + C$ is an antiderivative of $\cos x$.

d. Since

Equation:

$$\frac{d}{dx}(e^x) = e^x,$$

then $F(x) = e^x$ is an antiderivative of e^x . Therefore, every antiderivative of e^x is of the form $e^x + C$ for some constant C and every function of the form $e^x + C$ is an antiderivative of e^x .

Note:

Exercise:

Problem: Find all antiderivatives of $f(x) = \sin x$.

Solution:

$$-\cos x + C$$

Hint

What function has a derivative of $\sin x$?

Indefinite Integrals

We now look at the formal notation used to represent antiderivatives and examine some of their properties. These properties allow us to find antiderivatives of more complicated functions. Given a function f , we use the notation $f'(x)$ or $\frac{df}{dx}$ to denote the derivative of f . Here we introduce notation for antiderivatives. If F is an antiderivative of f , we say that $F(x) + C$ is the most general antiderivative of f and write

Equation:

$$\int f(x)dx = F(x) + C.$$

The symbol $\int$ is called an *integral sign*, and $\int f(x)dx$ is called the indefinite integral of f .

Note:

Definition

Given a function f , the **indefinite integral** of f , denoted

Equation:

$$\int f(x)dx,$$

is the most general antiderivative of f . If F is an antiderivative of f , then

Equation:

$$\int f(x)dx = F(x) + C.$$

The expression $f(x)$ is called the *integrand* and the variable x is the *variable of integration*.

Given the terminology introduced in this definition, the act of finding the antiderivatives of a function f is usually referred to as *integrating f* .

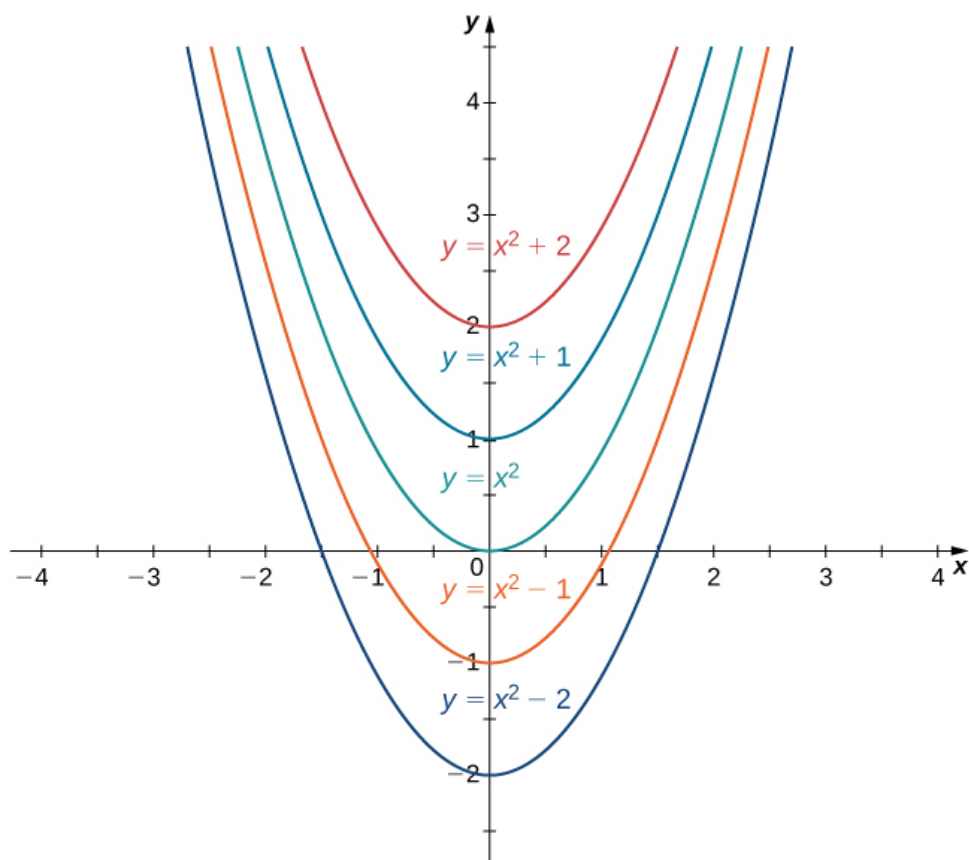
For a function f and an antiderivative F , the functions $F(x) + C$, where C is any real number, is often referred to as *the family of antiderivatives of f* . For example, since x^2 is an antiderivative of $2x$

and any antiderivative of $2x$ is of the form $x^2 + C$, we write

Equation:

$$\int 2x \, dx = x^2 + C.$$

The collection of all functions of the form $x^2 + C$, where C is any real number, is known as the *family of antiderivatives of $2x$* . [\[link\]](#) shows a graph of this family of antiderivatives.



The family of antiderivatives of $2x$ consists of all functions of the form $x^2 + C$, where C is any real number.

For some functions, evaluating indefinite integrals follows directly from properties of derivatives. For example, for $n \neq -1$,

Equation:

$$\int x^n \, dx = \frac{x^{n+1}}{n+1} + C,$$

which comes directly from

Equation:

$$\frac{d}{dx} \left(\frac{x^{n+1}}{n+1} \right) = (n+1) \frac{x^n}{n+1} = x^n.$$

This fact is known as *the power rule for integrals*.

Note:

Power Rule for Integrals

For $n \neq -1$,

Equation:

$$\int x^n dx = \frac{x^{n+1}}{n+1} + C.$$

Evaluating indefinite integrals for some other functions is also a straightforward calculation. The following table lists the indefinite integrals for several common functions. A more complete list appears in [Appendix B](#).

Differentiation Formula	Indefinite Integral
$\frac{d}{dx}(k) = 0$	$\int k dx = \int kx^0 dx = kx + C$
$\frac{d}{dx}(x^n) = nx^{n-1}$	$\int x^n dx = \frac{x^{n+1}}{n+1} + C$ for $n \neq -1$
$\frac{d}{dx}(\ln x) = \frac{1}{x}$	$\int \frac{1}{x} dx = \ln x + C$
$\frac{d}{dx}(e^x) = e^x$	$\int e^x dx = e^x + C$
$\frac{d}{dx}(\sin x) = \cos x$	$\int \cos x dx = \sin x + C$
$\frac{d}{dx}(\cos x) = -\sin x$	$\int \sin x dx = -\cos x + C$

Differentiation Formula	Indefinite Integral
$\frac{d}{dx}(\tan x) = \sec^2 x$	$\int \sec^2 x \, dx = \tan x + C$
$\frac{d}{dx}(\csc x) = -\csc x \cot x$	$\int \csc x \cot x \, dx = -\csc x + C$
$\frac{d}{dx}(\sec x) = \sec x \tan x$	$\int \sec x \tan x \, dx = \sec x + C$
$\frac{d}{dx}(\cot x) = -\csc^2 x$	$\int \csc^2 x \, dx = -\cot x + C$
$\frac{d}{dx}(\sin^{-1} x) = \frac{1}{\sqrt{1-x^2}}$	$\int \frac{1}{\sqrt{1-x^2}} = \sin^{-1} x + C$
$\frac{d}{dx}(\tan^{-1} x) = \frac{1}{1+x^2}$	$\int \frac{1}{1+x^2} \, dx = \tan^{-1} x + C$
$\frac{d}{dx}(\sec^{-1} x) = \frac{1}{x\sqrt{x^2-1}}$	$\int \frac{1}{x\sqrt{x^2-1}} \, dx = \sec^{-1} x + C$

Integration Formulas

From the definition of indefinite integral of f , we know

Equation:

$$\int f(x) \, dx = F(x) + C$$

if and only if F is an antiderivative of f . Therefore, when claiming that

Equation:

$$\int f(x) \, dx = F(x) + C$$

it is important to check whether this statement is correct by verifying that $F'(x) = f(x)$.

Example:

Exercise:

Problem:

Verifying an Indefinite Integral

Each of the following statements is of the form $\int f(x)dx = F(x) + C$. Verify that each statement is correct by showing that $F'(x) = f(x)$.

a. $\int (x + e^x)dx = \frac{x^2}{2} + e^x + C$

b. $\int xe^x dx = xe^x - e^x + C$

Solution:

a. Since

Equation:

$$\frac{d}{dx} \left(\frac{x^2}{2} + e^x + C \right) = x + e^x,$$

the statement

Equation:

$$\int (x + e^x)dx = \frac{x^2}{2} + e^x + C$$

is correct.

Note that we are verifying an indefinite integral for a sum. Furthermore, $\frac{x^2}{2}$ and e^x are antiderivatives of x and e^x , respectively, and the sum of the antiderivatives is an antiderivative of the sum. We discuss this fact again later in this section.

b. Using the product rule, we see that

Equation:

$$\frac{d}{dx}(xe^x - e^x + C) = e^x + xe^x - e^x = xe^x.$$

Therefore, the statement

Equation:

$$\int xe^x dx = xe^x - e^x + C$$

is correct.

Note that we are verifying an indefinite integral for a product. The antiderivative $xe^x - e^x$ is not a product of the antiderivatives. Furthermore, the product of antiderivatives, $x^2e^x/2$ is not an antiderivative of xe^x since

Equation:

$$\frac{d}{dx} \left(\frac{x^2 e^x}{2} \right) = x e^x + \frac{x^2 e^x}{2} \neq x e^x.$$

In general, the product of antiderivatives is not an antiderivative of a product.

Note:

Exercise:

Problem: Verify that $\int x \cos x \, dx = x \sin x + \cos x + C$.

Solution:

$$\frac{d}{dx} (x \sin x + \cos x + C) = \sin x + x \cos x - \sin x = x \cos x$$

Hint

Calculate $\frac{d}{dx} (x \sin x + \cos x + C)$.

In [\[link\]](#), we listed the indefinite integrals for many elementary functions. Let's now turn our attention to evaluating indefinite integrals for more complicated functions. For example, consider finding an antiderivative of a sum $f + g$. In [\[link\]](#)a, we showed that an antiderivative of the sum $x + e^x$ is given by the sum $\left(\frac{x^2}{2}\right) + e^x$ —that is, an antiderivative of a sum is given by a sum of antiderivatives. This result was not specific to this example. In general, if F and G are antiderivatives of any functions f and g , respectively, then

Equation:

$$\frac{d}{dx} (F(x) + G(x)) = F'(x) + G'(x) = f(x) + g(x).$$

Therefore, $F(x) + G(x)$ is an antiderivative of $f(x) + g(x)$ and we have

Equation:

$$\int (f(x) + g(x)) dx = F(x) + G(x) + C.$$

Similarly,

Equation:

$$\int (f(x) - g(x)) dx = F(x) - G(x) + C.$$

In addition, consider the task of finding an antiderivative of $kf(x)$, where k is any real number. Since
Equation:

$$\frac{d}{dx}(kf(x)) = k \frac{d}{dx}F(x) = kf'(x)$$

for any real number k , we conclude that

Equation:

$$\int kf(x)dx = kF(x) + C.$$

These properties are summarized next.

Note:

Properties of Indefinite Integrals

Let F and G be antiderivatives of f and g , respectively, and let k be any real number.

Sums and Differences

Equation:

$$\int (f(x) \pm g(x))dx = F(x) \pm G(x) + C$$

Constant Multiples

Equation:

$$\int kf(x)dx = kF(x) + C$$

From this theorem, we can evaluate any integral involving a sum, difference, or constant multiple of functions with antiderivatives that are known. Evaluating integrals involving products, quotients, or compositions is more complicated (see [\[link\]](#) for an example involving an antiderivative of a product.) We look at and address integrals involving these more complicated functions in [Introduction to Integration](#). In the next example, we examine how to use this theorem to calculate the indefinite integrals of several functions.

Example:

Exercise:

Problem:

Evaluating Indefinite Integrals

Evaluate each of the following indefinite integrals:

- a. $\int (5x^3 - 7x^2 + 3x + 4) dx$
- b. $\int \frac{x^2 + 4\sqrt[3]{x}}{x} dx$
- c. $\int \frac{4}{1+x^2} dx$
- d. $\int \tan x \cos x dx$

Solution:

- a. Using [\[link\]](#), we can integrate each of the four terms in the integrand separately. We obtain
Equation:

$$\int (5x^3 - 7x^2 + 3x + 4) dx = \int 5x^3 dx - \int 7x^2 dx + \int 3x dx + \int 4 dx.$$

From the second part of [\[link\]](#), each coefficient can be written in front of the integral sign, which gives

Equation:

$$\int 5x^3 dx - \int 7x^2 dx + \int 3x dx + \int 4 dx = 5 \int x^3 dx - 7 \int x^2 dx + 3 \int x dx + 4 \int 1 dx.$$

Using the power rule for integrals, we conclude that

Equation:

$$\int (5x^3 - 7x^2 + 3x + 4) dx = \frac{5}{4}x^4 - \frac{7}{3}x^3 + \frac{3}{2}x^2 + 4x + C.$$

- b. Rewrite the integrand as
Equation:

$$\frac{x^2 + 4\sqrt[3]{x}}{x} = \frac{x^2}{x} + \frac{4\sqrt[3]{x}}{x} = 0.$$

Then, to evaluate the integral, integrate each of these terms separately. Using the power rule, we have

Equation:

$$\begin{aligned} \int \left(x + \frac{4}{x^{2/3}} \right) dx &= \int x dx + 4 \int x^{-2/3} dx \\ &= \frac{1}{2}x^2 + 4 \frac{1}{(-\frac{2}{3})+1} x^{(-2/3)+1} + C \\ &= \frac{1}{2}x^2 + 12x^{1/3} + C. \end{aligned}$$

- c. Using [\[link\]](#), write the integral as

Equation:

$$4 \int \frac{1}{1+x^2} dx.$$

Then, use the fact that $\tan^{-1}(x)$ is an antiderivative of $\frac{1}{(1+x^2)}$ to conclude that

Equation:

$$\int \frac{4}{1+x^2} dx = 4 \tan^{-1}(x) + C.$$

d. Rewrite the integrand as

Equation:

$$\tan x \cos x = \frac{\sin x}{\cos x} \cos x = \sin x.$$

Therefore,

Equation:

$$\int \tan x \cos x = \int \sin x = -\cos x + C.$$

Note:

Exercise:

Problem: Evaluate $\int (4x^3 - 5x^2 + x - 7) dx$.

Solution:

$$x^4 - \frac{5}{3}x^3 + \frac{1}{2}x^2 - 7x + C$$

Hint

Integrate each term in the integrand separately, making use of the power rule.

Initial-Value Problems

We look at techniques for integrating a large variety of functions involving products, quotients, and compositions later in the text. Here we turn to one common use for antiderivatives that arises often in many applications: solving differential equations.

A *differential equation* is an equation that relates an unknown function and one or more of its derivatives. The equation

Equation:

$$\frac{dy}{dx} = f(x)$$

is a simple example of a differential equation. Solving this equation means finding a function y with a derivative f . Therefore, the solutions of [\[link\]](#) are the antiderivatives of f . If F is one antiderivative of f , every function of the form $y = F(x) + C$ is a solution of that differential equation. For example, the solutions of

Equation:

$$\frac{dy}{dx} = 6x^2$$

are given by

Equation:

$$y = \int 6x^2 dx = 2x^3 + C.$$

Sometimes we are interested in determining whether a particular solution curve passes through a certain point (x_0, y_0) —that is, $y(x_0) = y_0$. The problem of finding a function y that satisfies a differential equation

Equation:

$$\frac{dy}{dx} = f(x)$$

with the additional condition

Equation:

$$y(x_0) = y_0$$

is an example of an **initial-value problem**. The condition $y(x_0) = y_0$ is known as an *initial condition*. For example, looking for a function y that satisfies the differential equation

Equation:

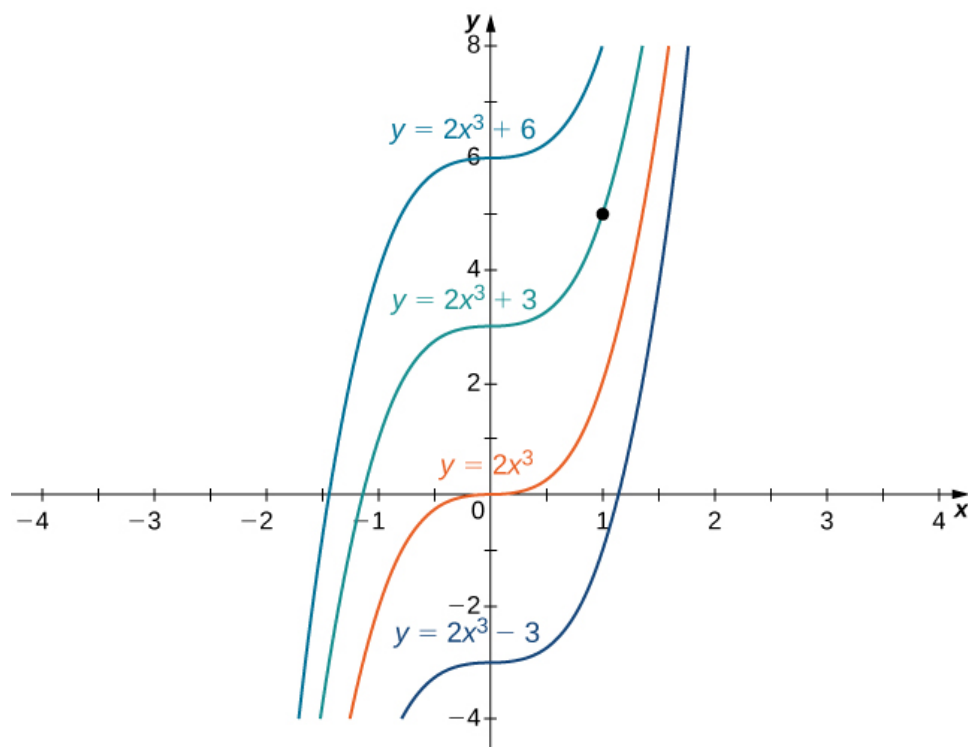
$$\frac{dy}{dx} = 6x^2$$

and the initial condition

Equation:

$$y(1) = 5$$

is an example of an initial-value problem. Since the solutions of the differential equation are $y = 2x^3 + C$, to find a function y that also satisfies the initial condition, we need to find C such that $y(1) = 2(1)^3 + C = 5$. From this equation, we see that $C = 3$, and we conclude that $y = 2x^3 + 3$ is the solution of this initial-value problem as shown in the following graph.



Some of the solution curves of the differential equation $\frac{dy}{dx} = 6x^2$ are displayed. The function $y = 2x^3 + 3$ satisfies the differential equation and the initial condition $y(1) = 5$.

Example:

Exercise:

Problem:

Solving an Initial-Value Problem

Solve the initial-value problem

Equation:

$$\frac{dy}{dx} = \sin x, y(0) = 5.$$

Solution:

First we need to solve the differential equation. If $\frac{dy}{dx} = \sin x$, then

Equation:

$$y = \int \sin(x) dx = -\cos x + C.$$

Next we need to look for a solution y that satisfies the initial condition. The initial condition $y(0) = 5$ means we need a constant C such that $-\cos x + C = 5$. Therefore,

Equation:

$$C = 5 + \cos(0) = 6.$$

The solution of the initial-value problem is $y = -\cos x + 6$.

Note:**Exercise:**

Problem: Solve the initial value problem $\frac{dy}{dx} = 3x^{-2}$, $y(1) = 2$.

Solution:

$$y = -\frac{3}{x} + 5$$

Hint

Find all antiderivatives of $f(x) = 3x^{-2}$.

Initial-value problems arise in many applications. Next we consider a problem in which a driver applies the brakes in a car. We are interested in how long it takes for the car to stop. Recall that the velocity function $v(t)$ is the derivative of a position function $s(t)$, and the acceleration $a(t)$ is the derivative of the velocity function. In earlier examples in the text, we could calculate the velocity from the position and then compute the acceleration from the velocity. In the next example we work the other way around. Given an acceleration function, we calculate the velocity function. We then use the velocity function to determine the position function.

Example:**Exercise:****Problem:****Decelerating Car**

A car is traveling at the rate of 88 ft/sec (60 mph) when the brakes are applied. The car begins decelerating at a constant rate of 15 ft/sec².

- a. How many seconds elapse before the car stops?
- b. How far does the car travel during that time?

Solution:

- a. First we introduce variables for this problem. Let t be the time (in seconds) after the brakes are first applied. Let $a(t)$ be the acceleration of the car (in feet per seconds squared) at time t . Let $v(t)$ be the velocity of the car (in feet per second) at time t . Let $s(t)$ be the car's position (in feet) beyond the point where the brakes are applied at time t . The car is traveling at a rate of 88 ft/sec. Therefore, the initial velocity is $v(0) = 88$ ft/sec. Since the car is decelerating, the acceleration is

Equation:

$$a(t) = -15 \text{ ft/s}^2.$$

The acceleration is the derivative of the velocity,

Equation:

$$v'(t) = -15.$$

Therefore, we have an initial-value problem to solve:

Equation:

$$v'(t) = -15, v(0) = 88.$$

Integrating, we find that

Equation:

$$v(t) = -15t + C.$$

Since $v(0) = 88$, $C = 88$. Thus, the velocity function is

Equation:

$$v(t) = -15t + 88.$$

To find how long it takes for the car to stop, we need to find the time t such that the velocity is zero. Solving $-15t + 88 = 0$, we obtain $t = \frac{88}{15}$ sec.

- b. To find how far the car travels during this time, we need to find the position of the car after $\frac{88}{15}$ sec. We know the velocity $v(t)$ is the derivative of the position $s(t)$. Consider the initial position to be $s(0) = 0$. Therefore, we need to solve the initial-value problem

Equation:

$$s'(t) = -15t + 88, s(0) = 0.$$

Integrating, we have

Equation:

$$s(t) = -\frac{15}{2}t^2 + 88t + C.$$

Since $s(0) = 0$, the constant is $C = 0$. Therefore, the position function is

Equation:

$$s(t) = -\frac{15}{2}t^2 + 88t.$$

After $t = \frac{88}{15}$ sec, the position is $s\left(\frac{88}{15}\right) \approx 258.133$ ft.

Note:

Exercise:

Problem:

Suppose the car is traveling at the rate of 44 ft/sec. How long does it take for the car to stop?
How far will the car travel?

Solution:

2.93 sec, 64.5 ft

Hint

$$v(t) = -15t + 44.$$

Key Concepts

- If F is an antiderivative of f , then every antiderivative of f is of the form $F(x) + C$ for some constant C .
- Solving the initial-value problem

Equation:

$$\frac{dy}{dx} = f(x), y(x_0) = y_0$$

requires us first to find the set of antiderivatives of f and then to look for the particular

antiderivative that also satisfies the initial condition.

For the following exercises, show that $F(x)$ are antiderivatives of $f(x)$.

Exercise:

Problem: $F(x) = 5x^3 + 2x^2 + 3x + 1, f(x) = 15x^2 + 4x + 3$

Solution:

$$F'(x) = 15x^2 + 4x + 3$$

Exercise:

Problem: $F(x) = x^2 + 4x + 1, f(x) = 2x + 4$

Exercise:

Problem: $F(x) = x^2 e^x, f(x) = e^x (x^2 + 2x)$

Solution:

$$F'(x) = 2xe^x + x^2 e^x$$

Exercise:

Problem: $F(x) = \cos x, f(x) = -\sin x$

Exercise:

Problem: $F(x) = e^x, f(x) = e^x$

Solution:

$$F'(x) = e^x$$

For the following exercises, find the antiderivative of the function.

Exercise:

Problem: $f(x) = \frac{1}{x^2} + x$

Exercise:

Problem: $f(x) = e^x - 3x^2 + \sin x$

Solution:

$$F(x) = e^x - x^3 - \cos(x) + C$$

Exercise:

Problem: $f(x) = e^x + 3x - x^2$

Exercise:

Problem: $f(x) = x - 1 + 4\sin(2x)$

Solution:

$$F(x) = \frac{x^2}{2} - x - 2\cos(2x) + C$$

For the following exercises, find the antiderivative $F(x)$ of each function $f(x)$.

Exercise:

Problem: $f(x) = 5x^4 + 4x^5$

Exercise:

Problem: $f(x) = x + 12x^2$

Solution:

$$F(x) = \frac{1}{2}x^2 + 4x^3 + C$$

Exercise:

Problem: $f(x) = \frac{1}{\sqrt{x}}$

Exercise:

Problem: $f(x) = (\sqrt{x})^3$

Solution:

$$F(x) = \frac{2}{5}(\sqrt{x})^5 + C$$

Exercise:

Problem: $f(x) = x^{1/3} + (2x)^{1/3}$

Exercise:

Problem: $f(x) = \frac{x^{1/3}}{x^{2/3}}$

Solution:

$$F(x) = \frac{3}{2}x^{2/3} + C$$

Exercise:

Problem: $f(x) = 2\sin(x) + \sin(2x)$

Exercise:

Problem: $f(x) = \sec^2(x) + 1$

Solution:

$$F(x) = x + \tan(x) + C$$

Exercise:

Problem: $f(x) = \sin x \cos x$

Exercise:

Problem: $f(x) = \sin^2(x) \cos(x)$

Solution:

$$F(x) = \frac{1}{3} \sin^3(x) + C$$

Exercise:

Problem: $f(x) = 0$

Exercise:

Problem: $f(x) = \frac{1}{2} \csc^2(x) + \frac{1}{x^2}$

Solution:

$$F(x) = -\frac{1}{2} \cot(x) - \frac{1}{x} + C$$

Exercise:

Problem: $f(x) = \csc x \cot x + 3x$

Exercise:

Problem: $f(x) = 4 \csc x \cot x - \sec x \tan x$

Solution:

$$F(x) = -\sec x - 4 \csc x + C$$

Exercise:

Problem: $f(x) = 8 \sec x (\sec x - 4 \tan x)$

Exercise:

Problem: $f(x) = \frac{1}{2}e^{-4x} + \sin x$

Solution:

$$F(x) = -\frac{1}{8}e^{-4x} - \cos x + C$$

For the following exercises, evaluate the integral.

Exercise:

Problem: $\int (-1)dx$

Exercise:

Problem: $\int \sin x \, dx$

Solution:

$$-\cos x + C$$

Exercise:

Problem: $\int (4x + \sqrt{x}) \, dx$

Exercise:

Problem: $\int \frac{3x^2 + 2}{x^2} \, dx$

Solution:

$$3x - \frac{2}{x} + C$$

Exercise:

Problem: $\int (\sec x \tan x + 4x) \, dx$

Exercise:

Problem: $\int (4\sqrt{x} + \sqrt[4]{x}) \, dx$

Solution:

$$\frac{8}{3}x^{3/2} + \frac{4}{5}x^{5/4} + C$$

Exercise:

Problem: $\int (x^{-1/3} - x^{2/3}) dx$

Exercise:

Problem: $\int \frac{14x^3 + 2x + 1}{x^3} dx$

Solution:

$$14x - \frac{2}{x} - \frac{1}{2x^2} + C$$

Exercise:

Problem: $\int (e^x + e^{-x}) dx$

For the following exercises, solve the initial value problem.

Exercise:

Problem: $f'(x) = x^{-3}, f(1) = 1$

Solution:

$$f(x) = -\frac{1}{2x^2} + \frac{3}{2}$$

Exercise:

Problem: $f'(x) = \sqrt{x} + x^2, f(0) = 2$

Exercise:

Problem: $f'(x) = \cos x + \sec^2(x), f\left(\frac{\pi}{4}\right) = 2 + \frac{\sqrt{2}}{2}$

Solution:

$$f(x) = \sin x + \tan x + 1$$

Exercise:

Problem: $f'(x) = x^3 - 8x^2 + 16x + 1, f(0) = 0$

Exercise:

Problem: $f'(x) = \frac{2}{x^2} - \frac{x^2}{2}, f(1) = 0$

Solution:

$$f(x) = -\frac{1}{6}x^3 - \frac{2}{x} + \frac{13}{6}$$

For the following exercises, find two possible functions f given the second- or third-order derivatives.

Exercise:

Problem: $f''(x) = x^2 + 2$

Exercise:

Problem: $f''(x) = e^{-x}$

Solution:

Answers may vary; one possible answer is $f(x) = e^{-x}$

Exercise:

Problem: $f''(x) = 1 + x$

Exercise:

Problem: $f'''(x) = \cos x$

Solution:

Answers may vary; one possible answer is $f(x) = -\sin x$

Exercise:

Problem: $f'''(x) = 8e^{-2x} - \sin x$

Exercise:

Problem:

A car is being driven at a rate of 40 mph when the brakes are applied. The car decelerates at a constant rate of 10 ft/sec^2 . How long before the car stops?

Solution:

5.867 sec

Exercise:

Problem: In the preceding problem, calculate how far the car travels in the time it takes to stop.

Exercise:

Problem:

You are merging onto the freeway, accelerating at a constant rate of 12 ft/sec^2 . How long does it take you to reach merging speed at 60 mph?

Solution:

7.333 sec

Exercise:

Problem: Based on the previous problem, how far does the car travel to reach merging speed?

Exercise:**Problem:**

A car company wants to ensure its newest model can stop in 8 sec when traveling at 75 mph. If we assume constant deceleration, find the value of deceleration that accomplishes this.

Solution:

13.75 ft/sec^2

Exercise:**Problem:**

A car company wants to ensure its newest model can stop in less than 450 ft when traveling at 60 mph. If we assume constant deceleration, find the value of deceleration that accomplishes this.

For the following exercises, find the antiderivative of the function, assuming $F'(0) = 0$.

Exercise:

Problem: [T] $f(x) = x^2 + 2$

Solution:

$$F(x) = \frac{1}{3}x^3 + 2x$$

Exercise:

Problem: [T] $f(x) = 4x - \sqrt{x}$

Exercise:

Problem: [T] $f(x) = \sin x + 2x$

Solution:

$$F(x) = x^2 - \cos x + 1$$

Exercise:

Problem: [T] $f(x) = e^x$

Exercise:

Problem: [T] $f(x) = \frac{1}{(x+1)^2}$

Solution:

$$F(x) = -\frac{1}{(x+1)} + 1$$

Exercise:

Problem: [T] $f(x) = e^{-2x} + 3x^2$

For the following exercises, determine whether the statement is true or false. Either prove it is true or find a counterexample if it is false.

Exercise:

Problem: If $f(x)$ is the antiderivative of $v(x)$, then $2f(x)$ is the antiderivative of $2v(x)$.

Solution:

True

Exercise:

Problem: If $f(x)$ is the antiderivative of $v(x)$, then $f(2x)$ is the antiderivative of $v(2x)$.

Exercise:

Problem: If $f(x)$ is the antiderivative of $v(x)$, then $f(x) + 1$ is the antiderivative of $v(x) + 1$.

Solution:

False

Exercise:

Problem: If $f(x)$ is the antiderivative of $v(x)$, then $(f(x))^2$ is the antiderivative of $(v(x))^2$.

Chapter Review Exercises

True or False? Justify your answer with a proof or a counterexample. Assume that $f(x)$ is continuous and differentiable unless stated otherwise.

Exercise:

Problem:

If $f(-1) = -6$ and $f(1) = 2$, then there exists at least one point $x \in [-1, 1]$ such that $f'(x) = 4$.

Solution:

True, by Mean Value Theorem

Exercise:

Problem: If $f'(c) = 0$, there is a maximum or minimum at $x = c$.

Exercise:**Problem:**

There is a function such that $f(x) < 0$, $f'(x) > 0$, and $f''(x) < 0$. (A graphical “proof” is acceptable for this answer.)

Solution:

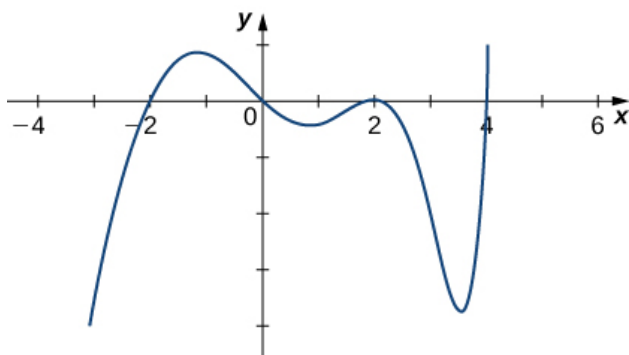
True

Exercise:**Problem:**

There is a function such that there is both an inflection point and a critical point for some value $x = a$.

Exercise:

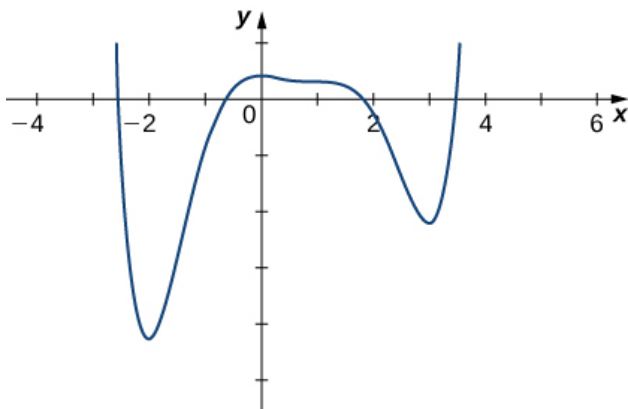
Problem: Given the graph of f' , determine where f is increasing or decreasing.

**Solution:**

Increasing: $(-2, 0) \cup (4, \infty)$, decreasing: $(-\infty, -2) \cup (0, 4)$

Exercise:

Problem: The graph of f is given below. Draw f' .



Exercise:

Problem: Find the linear approximation $L(x)$ to $y = x^2 + \tan(\pi x)$ near $x = \frac{1}{4}$.

Solution:

$$L(x) = \frac{17}{16} + \frac{1}{2}(1 + 4\pi)\left(x - \frac{1}{4}\right)$$

Exercise:

Problem: Find the differential of $y = x^2 - 5x - 6$ and evaluate for $x = 2$ with $dx = 0.1$.

Find the critical points and the local and absolute extrema of the following functions on the given interval.

Exercise:

Problem: $f(x) = x + \sin^2(x)$ over $[0, \pi]$

Solution:

Critical point: $x = \frac{3\pi}{4}$, absolute minimum: $x = 0$, absolute maximum: $x = \pi$

Exercise:

Problem: $f(x) = 3x^4 - 4x^3 - 12x^2 + 6$ over $[-3, 3]$

Determine over which intervals the following functions are increasing, decreasing, concave up, and concave down.

Exercise:

Problem: $x(t) = 3t^4 - 8t^3 - 18t^2$

Solution:

Increasing: $(-1, 0) \cup (3, \infty)$, decreasing: $(-\infty, -1) \cup (0, 3)$, concave up:
 $(-\infty, \frac{1}{3}(2 - \sqrt{13})) \cup (\frac{1}{3}(2 + \sqrt{13}), \infty)$, concave down:
 $(\frac{1}{3}(2 - \sqrt{13}), \frac{1}{3}(2 + \sqrt{13}))$

Exercise:

Problem: $y = x + \sin(\pi x)$

Exercise:

Problem: $g(x) = x - \sqrt{x}$

Solution:

Increasing: $(\frac{1}{4}, \infty)$, decreasing: $(0, \frac{1}{4})$, concave up: $(0, \infty)$, concave down: nowhere

Exercise:

Problem: $f(\theta) = \sin(3\theta)$

Evaluate the following limits.

Exercise:

Problem: $\lim_{x \rightarrow \infty} \frac{3x\sqrt{x^2+1}}{\sqrt{x^4-1}}$

Solution:

3

Exercise:

Problem: $\lim_{x \rightarrow \infty} \cos\left(\frac{1}{x}\right)$

Exercise:

Problem: $\lim_{x \rightarrow 1} \frac{x-1}{\sin(\pi x)}$

Solution:

$-\frac{1}{\pi}$

Exercise:

Problem: $\lim_{x \rightarrow \infty} (3x)^{1/x}$

Use Newton's method to find the first two iterations, given the starting point.

Exercise:

Problem: $y = x^3 + 1, x_0 = 0.5$

Solution:

$$x_1 = -1, x_2 = -1$$

Exercise:

Problem: $\frac{1}{x+1} = \frac{1}{2}, x_0 = 0$

Find the antiderivatives $F(x)$ of the following functions.

Exercise:

Problem: $g(x) = \sqrt{x} - \frac{1}{x^2}$

Solution:

$$F(x) = \frac{2x^{3/2}}{3} + \frac{1}{x} + C$$

Exercise:

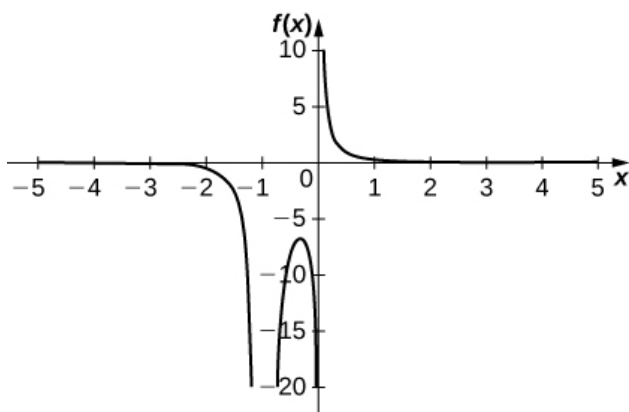
Problem: $f(x) = 2x + 6\cos x, F(\pi) = \pi^2 + 2$

Graph the following functions by hand. Make sure to label the inflection points, critical points, zeros, and asymptotes.

Exercise:

Problem: $y = \frac{1}{x(x+1)^2}$

Solution:



Inflection points: none; critical points: $x = -\frac{1}{3}$; zeros: none; vertical asymptotes: $x = -1$, $x = 0$; horizontal asymptote: $y = 0$

Exercise:

Problem: $y = x - \sqrt{4 - x^2}$

Exercise:

Problem:

A car is being compacted into a rectangular solid. The volume is decreasing at a rate of $2 \text{ m}^3/\text{sec}$. The length and width of the compactor are square, but the height is not the same length as the length and width. If the length and width walls move toward each other at a rate of 0.25 m/sec , find the rate at which the height is changing when the length and width are 2 m and the height is 1.5 m .

Solution:

The height is decreasing at a rate of 0.125 m/sec

Exercise:

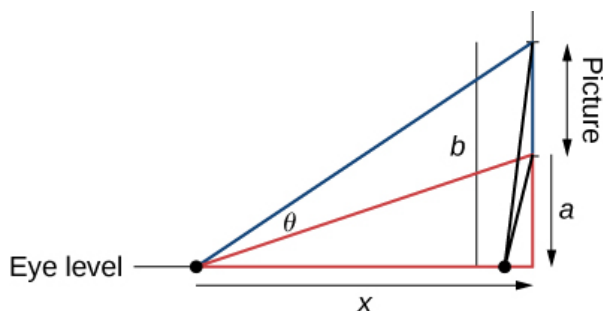
Problem:

A rocket is launched into space; its kinetic energy is given by $K(t) = \left(\frac{1}{2}\right)m(t)v(t)^2$, where K is the kinetic energy in joules, m is the mass of the rocket in kilograms, and v is the velocity of the rocket in meters/second. Assume the velocity is increasing at a rate of 15 m/sec^2 and the mass is decreasing at a rate of 10 kg/sec because the fuel is being burned. At what rate is the rocket's kinetic energy changing when the mass is 2000 kg and the velocity is 5000 m/sec ? Give your answer in mega-Joules (MJ), which is equivalent to 10^6 J .

Exercise:

Problem:

The famous Regiomontanus' problem for angle maximization was proposed during the 15th century. A painting hangs on a wall with the bottom of the painting a distance a feet above eye level, and the top b feet above eye level. What distance x (in feet) from the wall should the viewer stand to maximize the angle subtended by the painting, θ ?



Solution:

$$x = \sqrt{ab} \text{ feet}$$

Exercise:**Problem:**

An airline sells tickets from Tokyo to Detroit for \$1200. There are 500 seats available and a typical flight books 350 seats. For every \$10 decrease in price, the airline observes an additional five seats sold. What should the fare be to maximize profit? How many passengers would be onboard?

Glossary

antiderivative

a function F such that $F'(x) = f(x)$ for all x in the domain of f is an antiderivative of f

indefinite integral

the most general antiderivative of $f(x)$ is the indefinite integral of f ; we use the notation

$\int f(x)dx$ to denote the indefinite integral of f

initial value problem

a problem that requires finding a function y that satisfies the differential equation $\frac{dy}{dx} = f(x)$ together with the initial condition $y(x_0) = y_0$

Integrals Involving Exponential and Logarithmic Functions

- Integrate functions involving exponential functions.
- Integrate functions involving logarithmic functions.

Exponential and logarithmic functions are used to model population growth, cell growth, and financial growth, as well as depreciation, radioactive decay, and resource consumption, to name only a few applications. In this section, we explore integration involving exponential and logarithmic functions.

Integrals of Exponential Functions

The exponential function is perhaps the most efficient function in terms of the operations of calculus. The exponential function, $y = e^x$, is its own derivative and its own integral.

Note:

Rule: Integrals of Exponential Functions

Exponential functions can be integrated using the following formulas.

Equation:

$$\begin{aligned}\int e^x dx &= e^x + C \\ \int a^x dx &= \frac{a^x}{\ln a} + C\end{aligned}$$

Example:**Exercise:****Problem:**

Finding an Antiderivative of an Exponential Function

Find the antiderivative of the exponential function e^{-x} .

Solution:

Use substitution, setting $u = -x$, and then $du = -1dx$. Multiply the du equation by -1 , so you now have $-du = dx$. Then,

Equation:

$$\begin{aligned}\int e^{-x} dx &= -\int e^u du \\ &= -e^u + C \\ &= -e^{-x} + C.\end{aligned}$$

Note:

Exercise:

Problem:

Find the antiderivative of the function using substitution: $x^2 e^{-2x^3}$.

Solution:

$$\int x^2 e^{-2x^3} dx = -\frac{1}{6} e^{-2x^3} + C$$

Hint

Let u equal the exponent on e .

A common mistake when dealing with exponential expressions is treating the exponent on e the same way we treat exponents in polynomial expressions. We cannot use the power rule for the exponent on e . This can

be especially confusing when we have both exponentials and polynomials in the same expression, as in the previous checkpoint. In these cases, we should always double-check to make sure we're using the right rules for the functions we're integrating.

Example:

Exercise:

Problem:

Square Root of an Exponential Function

Find the antiderivative of the exponential function $e^x \sqrt{1 + e^x}$.

Solution:

First rewrite the problem using a rational exponent:

Equation:

$$\int e^x \sqrt{1 + e^x} dx = \int e^x (1 + e^x)^{1/2} dx.$$

Using substitution, choose $u = 1 + e^x$. $u = 1 + e^x$. Then, $du = e^x dx$. We have ([link](#))

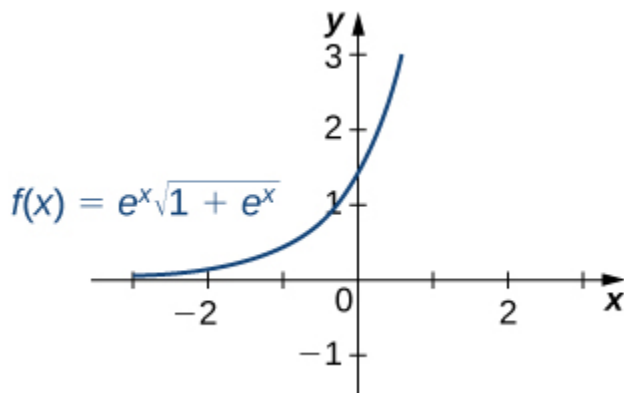
Equation:

$$\int e^x (1 + e^x)^{1/2} dx = \int u^{1/2} du.$$

Then

Equation:

$$\int u^{1/2} du = \frac{u^{3/2}}{3/2} + C = \frac{2}{3} u^{3/2} + C = \frac{2}{3} (1 + e^x)^{3/2} + C.$$



The graph shows an exponential function times the square root of an exponential function.

Note:

Exercise:

Problem: Find the antiderivative of $e^x(3e^x - 2)^2$.

Solution:

$$\int e^x(3e^x - 2)^2 dx = \frac{1}{9}(3e^x - 2)^3$$

Hint

Let $u = 3e^x - 2$ $u = 3e^x - 2$.

Example:

Exercise:

Problem:**Using Substitution with an Exponential Function**

Use substitution to evaluate the indefinite integral $\int 3x^2 e^{2x^3} dx$.

Solution:

Here we choose to let u equal the expression in the exponent on e . Let $u = 2x^3$ and $du = 6x^2 dx$. Again, du is off by a constant multiplier; the original function contains a factor of $3x^2$, not $6x^2$. Multiply both sides of the equation by $\frac{1}{2}$ so that the integrand in u equals the integrand in x . Thus,

Equation:

$$\int 3x^2 e^{2x^3} dx = \frac{1}{2} \int e^u du.$$

Integrate the expression in u and then substitute the original expression in x back into the u integral:

Equation:

$$\frac{1}{2} \int e^u du = \frac{1}{2} e^u + C = \frac{1}{2} e^{2x^3} + C.$$

Note:**Exercise:**

Problem: Evaluate the indefinite integral $\int 2x^3 e^{x^4} dx$.

Solution:

$$\int 2x^3 e^{x^4} dx = \frac{1}{2} e^{x^4}$$

Hint

Let $u = x^4$.

As mentioned at the beginning of this section, exponential functions are used in many real-life applications. The number e is often associated with compounded or accelerating growth, as we have seen in earlier sections about the derivative. Although the derivative represents a rate of change or a growth rate, the integral represents the total change or the total growth. Let's look at an example in which integration of an exponential function solves a common business application.

A price–demand function tells us the relationship between the quantity of a product demanded and the price of the product. In general, price decreases as quantity demanded increases. The marginal price–demand function is the derivative of the price–demand function and it tells us how fast the price changes at a given level of production. These functions are used in business to determine the price–elasticity of demand, and to help companies determine whether changing production levels would be profitable.

Example:**Exercise:****Problem:****Finding a Price–Demand Equation**

Find the price–demand equation for a particular brand of toothpaste at a supermarket chain when the demand is 50 tubes per week at \$2.35 per tube, given that the marginal price–demand function, $p'(x)$, for x number of tubes per week, is given as

Equation:

$$p'(x) = -0.015e^{-0.01x}.$$

If the supermarket chain sells 100 tubes per week, what price should it set?

Solution:

To find the price–demand equation, integrate the marginal price–demand function. First find the antiderivative, then look at the particulars. Thus,

Equation:

$$\begin{aligned} p(x) &= \int -0.015e^{-0.01x} dx \\ &= -0.015 \int e^{-0.01x} dx. \end{aligned}$$

Using substitution, let $u = -0.01x$ and $du = -0.01dx$. Then, divide both sides of the du equation by -0.01 . This gives

Equation:

$$\begin{aligned} \frac{-0.015}{-0.01} \int e^u du &= 1.5 \int e^u du \\ &= 1.5e^u + C \\ &= 1.5e^{-0.01x} + C. \end{aligned}$$

The next step is to solve for C . We know that when the price is \$2.35 per tube, the demand is 50 tubes per week. This means

Equation:

$$\begin{aligned} p(50) &= 1.5e^{-0.01(50)} + C \\ &= 2.35. \end{aligned}$$

Now, just solve for C :

Equation:

$$\begin{aligned}C &= 2.35 - 1.5e^{-0.5} \\&= 2.35 - 0.91 \\&= 1.44.\end{aligned}$$

Thus,

Equation:

$$p(x) = 1.5e^{-0.01x} + 1.44.$$

If the supermarket sells 100 tubes of toothpaste per week, the price would be

Equation:

$$p(100) = 1.5e^{-0.01(100)} + 1.44 = 1.5e^{-1} + 1.44 \approx 1.99.$$

The supermarket should charge \$1.99 per tube if it is selling 100 tubes per week.

Example:

Exercise:

Problem:

Evaluating a Definite Integral Involving an Exponential Function

Evaluate the definite integral $\int_1^2 e^{1-x} dx$.

Solution:

Again, substitution is the method to use. Let $u = 1 - x$, so $du = -1dx$ or $-du = dx$. Then $\int e^{1-x} dx = -\int e^u du$. Next, change the limits of integration. Using the equation $u = 1 - x$, we have

Equation:

$$u = 1 - (1) = 0$$

$$u = 1 - (2) = -1.$$

The integral then becomes

Equation:

$$\int_1^2 e^{1-x} dx = -\int_0^{-1} e^u du$$

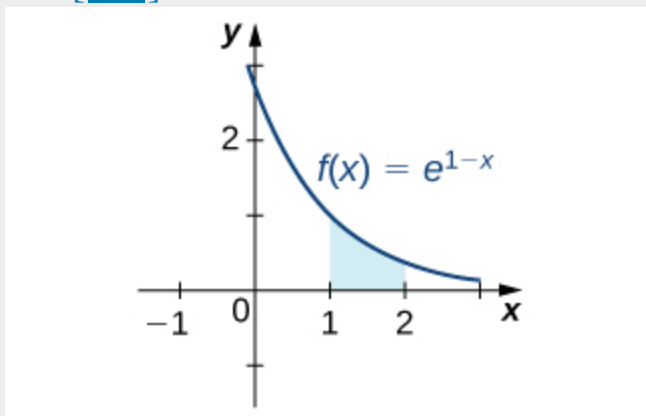
$$= \int_{-1}^0 e^u du$$

$$= e^u \Big|_{-1}^0$$

$$= e^0 - (e^{-1})$$

$$= -e^{-1} + 1.$$

See [\[link\]](#).



The indicated area can be calculated by evaluating a definite integral using substitution.

Note:

Exercise:

Problem: Evaluate $\int_0^2 e^{2x} dx$.

Solution:

$$\frac{1}{2} \int_0^4 e^u du = \frac{1}{2} (e^4 - 1)$$

Hint

Let $u = 2x$.

Example:

Exercise:

Problem:

Growth of Bacteria in a Culture

Suppose the rate of growth of bacteria in a Petri dish is given by $q(t) = 3^t$, where t is given in hours and $q(t)$ is given in thousands of bacteria per hour. If a culture starts with 10,000 bacteria, find a function $Q(t)$ that gives the number of bacteria in the Petri dish at any time t . How many bacteria are in the dish after 2 hours?

Solution:

We have

Equation:

$$Q(t) = \int 3^t dt = \frac{3^t}{\ln 3} + C.$$

Then, at $t = 0$ we have $Q(0) = 10 = \frac{1}{\ln 3} + C$, so $C \approx 9.090$ and we get

Equation:

$$Q(t) = \frac{3^t}{\ln 3} + 9.090.$$

At time $t = 2$, we have

Equation:

$$Q(2) = \frac{3^2}{\ln 3} + 9.090$$

Equation:

$$= 17.282.$$

After 2 hours, there are 17,282 bacteria in the dish.

Note:

Exercise:

Problem:

From [\[link\]](#), suppose the bacteria grow at a rate of $q(t) = 2^t$. Assume the culture still starts with 10,000 bacteria. Find $Q(t)$. How many bacteria are in the dish after 3 hours?

Solution:

$Q(t) = \frac{2^t}{\ln 2} + 8.557$. There are 20,099 bacteria in the dish after 3 hours.

Hint

Use the procedure from [\[link\]](#) to solve the problem.

Example:**Exercise:****Problem:****Fruit Fly Population Growth**

Suppose a population of fruit flies increases at a rate of $g(t) = 2e^{0.02t}$, in flies per day. If the initial population of fruit flies is 100 flies, how many flies are in the population after 10 days?

Solution:

Let $G(t)$ represent the number of flies in the population at time t . Applying the net change theorem, we have

Equation:

$$\begin{aligned}
 G(10) &= G(0) + \int_0^{10} 2e^{0.02t} dt \\
 &= 100 + \left[\frac{2}{0.02} e^{0.02t} \right]_0^{10} \\
 &= 100 + \left[100e^{0.02t} \right]_0^{10} \\
 &= 100 + 100e^{0.2} - 100 \\
 &\approx 122.
 \end{aligned}$$

There are 122 flies in the population after 10 days.

Note:

Exercise:

Problem:

Suppose the rate of growth of the fly population is given by $g(t) = e^{0.01t}$, and the initial fly population is 100 flies. How many flies are in the population after 15 days?

Solution:

There are 116 flies.

Hint

Use the process from [\[link\]](#) to solve the problem.

Example:

Exercise:

Problem:

Evaluating a Definite Integral Using Substitution

Evaluate the definite integral using substitution: $\int_1^2 \frac{e^{1/x}}{x^2} dx$.

Solution:

This problem requires some rewriting to simplify applying the properties. First, rewrite the exponent on e as a power of x , then bring the x^2 in the denominator up to the numerator using a negative exponent. We have

Equation:

$$\int_1^2 \frac{e^{1/x}}{x^2} dx = \int_1^2 e^{x^{-1}} x^{-2} dx.$$

Let $u = x^{-1}$, the exponent on e . Then

Equation:

$$\begin{aligned} du &= -x^{-2} dx \\ -du &= x^{-2} dx. \end{aligned}$$

Bringing the negative sign outside the integral sign, the problem now reads

Equation:

$$- \int e^u du.$$

Next, change the limits of integration:

Equation:

$$u = (1)^{-1} = 1$$

$$u = (2)^{-1} = \frac{1}{2}.$$

Notice that now the limits begin with the larger number, meaning we must multiply by -1 and interchange the limits. Thus,

Equation:

$$\begin{aligned}
 - \int_1^{1/2} e^u du &= \int_{1/2}^1 e^u du \\
 &= e^u \Big|_{1/2}^1 \\
 &= e - e^{1/2} \\
 &= e - \sqrt{e}.
 \end{aligned}$$

Note:

Exercise:

Problem:

Evaluate the definite integral using substitution: $\int_1^2 \frac{1}{x^3} e^{4x^{-2}} dx.$

Solution:

$$\int_1^2 \frac{1}{x^3} e^{4x^{-2}} dx = \frac{1}{8} [e^4 - e]$$

Hint

Let $u = 4x^{-2}.$

Integrals Involving Logarithmic Functions

Integrating functions of the form $f(x) = x^{-1}$ result in the absolute value of the natural log function, as shown in the following rule. Integral formulas for other logarithmic functions, such as $f(x) = \ln x$ and $f(x) = \log_a x$, are also included in the rule.

Note:

Rule: Integration Formulas Involving Logarithmic Functions

The following formulas can be used to evaluate integrals involving logarithmic functions.

Equation:

$$\begin{aligned}\int x^{-1} dx &= \ln |x| + C \\ \int \ln x \, dx &= x \ln x - x + C = x(\ln x - 1) + C \\ \int \log_a x \, dx &= \frac{x}{\ln a} (\ln x - 1) + C\end{aligned}$$

Example:

Exercise:

Problem:

Finding an Antiderivative Involving $\ln x$

Find the antiderivative of the function $\frac{3}{x-10}$.

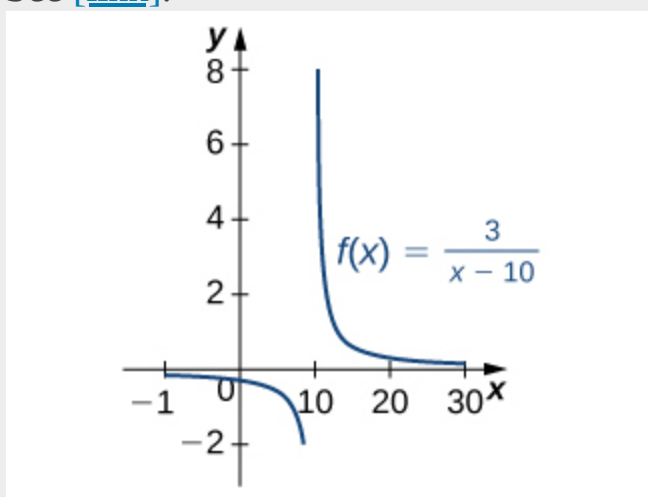
Solution:

First factor the 3 outside the integral symbol. Then use the u^{-1} rule. Thus,

Equation:

$$\begin{aligned}\int \frac{3}{x-10} dx &= 3 \int \frac{1}{x-10} dx \\ &= 3 \int \frac{du}{u} \\ &= 3 \ln |u| + C \\ &= 3 \ln |x-10| + C, x \neq 10.\end{aligned}$$

See [\[link\]](#).



The domain of this function is
 $x \neq 10$.

Note:

Exercise:

Problem: Find the antiderivative of $\frac{1}{x+2}$.

Solution:

$$\ln |x + 2| + C$$

Hint

Follow the pattern from [\[link\]](#) to solve the problem.

Example:**Exercise:****Problem:****Finding an Antiderivative of a Rational Function**

Find the antiderivative of $\frac{2x^3+3x}{x^4+3x^2}$.

Solution:

This can be rewritten as $\int (2x^3 + 3x)(x^4 + 3x^2)^{-1} dx$. Use substitution. Let $u = x^4 + 3x^2$, then $du = 4x^3 + 6x$. Alter du by factoring out the 2. Thus,

Equation:

$$\begin{aligned} du &= (4x^3 + 6x)dx \\ &= 2(2x^3 + 3x)dx \\ \frac{1}{2} du &= (2x^3 + 3x)dx. \end{aligned}$$

Rewrite the integrand in u :

Equation:

$$\int (2x^3 + 3x)(x^4 + 3x^2)^{-1} dx = \frac{1}{2} \int u^{-1} du.$$

Then we have

Equation:

$$\begin{aligned}\frac{1}{2} \int u^{-1} du &= \frac{1}{2} \ln |u| + C \\ &= \frac{1}{2} \ln |x^4 + 3x^2| + C.\end{aligned}$$

Example:

Exercise:

Problem:

Finding an Antiderivative of a Logarithmic Function

Find the antiderivative of the log function $\log_2 x$.

Solution:

Follow the format in the formula listed in the rule on integration formulas involving logarithmic functions. Based on this format, we have

Equation:

$$\int \log_2 x dx = \frac{x}{\ln 2} (\ln x - 1) + C.$$

Note:

Exercise:

Problem: Find the antiderivative of $\log_3 x$.

Solution:

$$\frac{x}{\ln 3} (\ln x - 1) + C$$

Hint

Follow [\[link\]](#) and refer to the rule on integration formulas involving logarithmic functions.

[\[link\]](#) is a definite integral of a trigonometric function. With trigonometric functions, we often have to apply a trigonometric property or an identity before we can move forward. Finding the right form of the integrand is usually the key to a smooth integration.

Example:**Exercise:****Problem:****Evaluating a Definite Integral**

Find the definite integral of $\int_0^{\pi/2} \frac{\sin x}{1 + \cos x} dx$.

Solution:

We need substitution to evaluate this problem. Let $u = 1 + \cos x$, so $du = -\sin x \, dx$. Rewrite the integral in terms of u , changing the limits of integration as well. Thus,

Equation:

$$u = 1 + \cos(0) = 2$$

$$u = 1 + \cos\left(\frac{\pi}{2}\right) = 1.$$

Then

Equation:

$$\begin{aligned}\int_0^{\pi/2} \frac{\sin x}{1 + \cos x} &= - \int_2^1 u^{-1} du \\ &= \int_1^2 u^{-1} du \\ &= \ln |u| \Big|_1^2 \\ &= [\ln 2 - \ln 1] \\ &= \ln 2.\end{aligned}$$

Key Concepts

- Exponential and logarithmic functions arise in many real-world applications, especially those involving growth and decay.
- Substitution is often used to evaluate integrals involving exponential functions or logarithms.

Key Equations

- **Integrals of Exponential Functions**

$$\int e^x dx = e^x + C$$

$$\int a^x dx = \frac{a^x}{\ln a} + C$$

- **Integration Formulas Involving Logarithmic Functions**

$$\int x^{-1} dx = \ln |x| + C$$

$$\int \ln x \, dx = x \ln x - x + C = x (\ln x - 1) + C$$

$$\int \log_a x \, dx = \frac{x}{\ln a} (\ln x - 1) + C$$

In the following exercises, compute each indefinite integral.

Exercise:

Problem: $\int e^{2x} dx$

Exercise:

Problem: $\int e^{-3x} dx$

Solution:

$$-\frac{1}{3} e^{-3x} + C$$

Exercise:

Problem: $\int 2^x dx$

Exercise:

Problem: $\int 3^{-x} dx$

Solution:

$$-\frac{3^{-x}}{\ln 3} + C$$

Exercise:

Problem: $\int \frac{1}{2x} dx$

Exercise:

Problem: $\int \frac{2}{x} dx$

Solution:

$$\ln(x^2) + C$$

Exercise:

Problem: $\int \frac{1}{x^2} dx$

Exercise:

Problem: $\int \frac{1}{\sqrt{x}} dx$

Solution:

$$2\sqrt{x} + C$$

In the following exercises, find each indefinite integral by using appropriate substitutions.

Exercise:

Problem: $\int \frac{\ln x}{x} dx$

Exercise:

Problem: $\int \frac{dx}{x(\ln x)^2}$

Solution:

$$-\frac{1}{\ln x} + C$$

Exercise:

Problem: $\int \frac{dx}{x \ln x} \quad (x > 1)$

Exercise:

Problem: $\int \frac{dx}{x \ln x \ln(\ln x)}$

Solution:

$$\ln(\ln(\ln x)) + C$$

Exercise:

Problem: $\int \tan \theta \, d\theta$

Exercise:

Problem: $\int \frac{\cos x - x \sin x}{x \cos x} dx$

Solution:

$$\ln(x \cos x) + C$$

Exercise:

Problem: $\int \frac{\ln(\sin x)}{\tan x} dx$

Exercise:

Problem: $\int \ln(\cos x) \tan x dx$

Solution:

$$-\frac{1}{2} (\ln(\cos(x)))^2 + C$$

Exercise:

Problem: $\int x e^{-x^2} dx$

Exercise:

Problem: $\int x^2 e^{-x^3} dx$

Solution:

$$\frac{-e^{-x^3}}{3} + C$$

Exercise:

Problem: $\int e^{\sin x} \cos x dx$

Exercise:

Problem: $\int e^{\tan x} \sec^2 x dx$

Solution:

$$e^{\tan x} + C$$

Exercise:

Problem: $\int e^{\ln x} \frac{dx}{x}$

Exercise:

Problem: $\int \frac{e^{\ln(1-t)}}{1-t} dt$

Solution:

$$t + C$$

In the following exercises, verify by differentiation that

$\int \ln x \, dx = x (\ln x - 1) + C$, then use appropriate changes of variables to compute the integral.

Exercise:

Problem: $\int x \ln x \, dx$ (*Hint:* $\int x \ln x \, dx = \frac{1}{2} \int x \ln(x^2) \, dx$; $x > 0$)

Exercise:

Problem: $\int x^2 \ln^2 x \, dx$

Solution:

$$\frac{1}{9} x^3 (\ln(x^3) - 1) + C$$

Exercise:

Problem: $\int \frac{\ln x}{x^2} dx$ (*Hint:* Set $u = \frac{1}{x}$.)

Exercise:

Problem: $\int \frac{\ln x}{\sqrt{x}} dx$ (*Hint:* Set $u = \sqrt{x}$.)

Solution:

$$2\sqrt{x}(\ln x - 2) + C$$

Exercise:

Problem:

Write an integral to express the area under the graph of $y = \frac{1}{t}$ from $t = 1$ to e^x and evaluate the integral.

Exercise:

Problem:

Write an integral to express the area under the graph of $y = e^t$ between $t = 0$ and $t = \ln x$, and evaluate the integral.

Solution:

$$\int_0^{\ln x} e^t dt = e^t \Big|_0^{\ln x} = e^{\ln x} - e^0 = x - 1$$

In the following exercises, use appropriate substitutions to express the trigonometric integrals in terms of compositions with logarithms.

Exercise:

Problem: $\int \tan(2x) dx$

Exercise:

Problem: $\int \frac{\sin(3x) - \cos(3x)}{\sin(3x) + \cos(3x)} dx$

Solution:

$$-\frac{1}{3} \ln(\sin(3x) + \cos(3x))$$

Exercise:

Problem: $\int \frac{x \sin(x^2)}{\cos(x^2)} dx$

Exercise:

Problem: $\int x \csc(x^2) dx$

Solution:

$$-\frac{1}{2} \ln |\csc(x^2) + \cot(x^2)| + C$$

Exercise:

Problem: $\int \ln(\cos x) \tan x dx$

Exercise:

Problem: $\int \ln(\csc x) \cot x dx$

Solution:

$$-\frac{1}{2} (\ln(\csc x))^2 + C$$

Exercise:

Problem: $\int \frac{e^x - e^{-x}}{e^x + e^{-x}} dx$

In the following exercises, evaluate the definite integral.

Exercise:

Problem: $\int_1^2 \frac{1 + 2x + x^2}{3x + 3x^2 + x^3} dx$

Solution:

$$\frac{1}{3} \ln \left(\frac{26}{7} \right)$$

Exercise:

Problem: $\int_0^{\pi/4} \tan x \, dx$

Exercise:

Problem: $\int_0^{\pi/3} \frac{\sin x - \cos x}{\sin x + \cos x} dx$

Solution:

$$\ln \left(\sqrt{3} - 1 \right)$$

Exercise:

Problem: $\int_{\pi/6}^{\pi/2} \csc x \, dx$

Exercise:

Problem: $\int_{\pi/4}^{\pi/3} \cot x dx$

Solution:

$$\frac{1}{2} \ln \frac{3}{2}$$

In the following exercises, integrate using the indicated substitution.

Exercise:

Problem: $\int \frac{x}{x-100} dx; u = x - 100$

Exercise:

Problem: $\int \frac{y-1}{y+1} dy; u = y + 1$

Solution:

$$y - 2 \ln |y + 1| + C$$

Exercise:

Problem: $\int \frac{1-x^2}{3x-x^3} dx; u = 3x - x^3$

Exercise:

Problem: $\int \frac{\sin x + \cos x}{\sin x - \cos x} dx; u = \sin x - \cos x$

Solution:

$$\ln |\sin x - \cos x| + C$$

Exercise:

Problem: $\int e^{2x} \sqrt{1 - e^{2x}} dx; u = e^{2x}$

Exercise:

Problem: $\int \ln(x) \frac{\sqrt{1 - (\ln x)^2}}{x} dx; u = \ln x$

Solution:

$$-\frac{1}{3} (1 - (\ln x)^2)^{3/2} + C$$

In the following exercises, does the right-endpoint approximation overestimate or underestimate the exact area? Calculate the right endpoint estimate R_{50} and solve for the exact area.

Exercise:

Problem: [T] $y = e^x$ over $[0, 1]$

Exercise:

Problem: [T] $y = e^{-x}$ over $[0, 1]$

Solution:

Exact solution: $\frac{e-1}{e}$, $R_{50} = 0.6258$. Since f is decreasing, the right endpoint estimate underestimates the area.

Exercise:

Problem: [T] $y = \ln(x)$ over $[1, 2]$

Exercise:

Problem: [T] $y = \frac{x+1}{x^2+2x+6}$ over $[0, 1]$

Solution:

Exact solution: $\frac{2\ln(3)-\ln(6)}{2}$, $R_{50} = 0.2033$. Since f is increasing, the right endpoint estimate overestimates the area.

Exercise:

Problem: [T] $y = 2^x$ over $[-1, 0]$

Exercise:

Problem: [T] $y = -2^{-x}$ over $[0, 1]$

Solution:

Exact solution: $-\frac{1}{\ln(4)}$, $R_{50} = -0.7164$. Since f is increasing, the right endpoint estimate overestimates the area (the actual area is a larger negative number).

In the following exercises, $f(x) \geq 0$ for $a \leq x \leq b$. Find the area under the graph of $f(x)$ between the given values a and b by integrating.

Exercise:

Problem: $f(x) = \frac{\log_{10}(x)}{x}$; $a = 10, b = 100$

Exercise:

Problem: $f(x) = \frac{\log_2(x)}{x}$; $a = 32, b = 64$

Solution:

$$\frac{11}{2} \ln 2$$

Exercise:

Problem: $f(x) = 2^{-x}; a = 1, b = 2$

Exercise:

Problem: $f(x) = 2^{-x}; a = 3, b = 4$

Solution:

$$\frac{1}{\ln(65,536)}$$

Exercise:**Problem:**

Find the area under the graph of the function $f(x) = xe^{-x^2}$ between $x = 0$ and $x = 5$.

Exercise:**Problem:**

Compute the integral of $f(x) = xe^{-x^2}$ and find the smallest value of N such that the area under the graph $f(x) = xe^{-x^2}$ between $x = N$ and $x = N + 1$ is, at most, 0.01.

Solution:

$\int_N^{N+1} xe^{-x^2} dx = \frac{1}{2} (e^{-N^2} - e^{-(N+1)^2})$. The quantity is less than 0.01 when $N = 2$.

Exercise:**Problem:**

Find the limit, as N tends to infinity, of the area under the graph of $f(x) = xe^{-x^2}$ between $x = 0$ and $x = 5$.

Exercise:

Problem: Show that $\int_a^b \frac{dt}{t} = \int_{1/b}^{1/a} \frac{dt}{t}$ when $0 < a \leq b$.

Solution:

$$\int_a^b \frac{dx}{x} = \ln(b) - \ln(a) = \ln\left(\frac{1}{a}\right) - \ln\left(\frac{1}{b}\right) = \int_{1/b}^{1/a} \frac{dx}{x}$$

Exercise:**Problem:**

Suppose that $f(x) > 0$ for all x and that f and g are differentiable. Use the identity $f^g = e^{g \ln f}$ and the chain rule to find the derivative of f^g .

Exercise:**Problem:**

Use the previous exercise to find the antiderivative of

$$h(x) = x^x (1 + \ln x) \text{ and evaluate } \int_2^3 x^x (1 + \ln x) dx.$$

Solution:

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Exercise:**Problem:**

Show that if $c > 0$, then the integral of $1/x$ from ac to bc ($0 < a < b$) is the same as the integral of $1/x$ from a to b .

The following exercises are intended to derive the fundamental properties of the natural log starting from the *definition* $\ln(x) = \int_1^x \frac{dt}{t}$, using

properties of the definite integral and making no further assumptions.

Exercise:

Problem:

Use the identity $\ln(x) = \int_1^x \frac{dt}{t}$ to derive the identity $\ln\left(\frac{1}{x}\right) = -\ln x$.

Solution:

We may assume that $x > 1$, so $\frac{1}{x} < 1$. Then, $\int_1^{1/x} \frac{dt}{t}$. Now make the substitution $u = \frac{1}{t}$, so $du = -\frac{dt}{t^2}$ and $\frac{du}{u} = -\frac{dt}{t}$, and change endpoints: $\int_1^{1/x} \frac{dt}{t} = -\int_1^x \frac{du}{u} = -\ln x$.

Exercise:

Problem:

Use a change of variable in the integral $\int_1^{xy} \frac{1}{t} dt$ to show that $\ln xy = \ln x + \ln y$ for $x, y > 0$.

Exercise:

Problem:

Use the identity $\ln x = \int_1^x \frac{dt}{t}$ to show that $\ln(x)$ is an increasing function of x on $[0, \infty)$, and use the previous exercises to show that the range of $\ln(x)$ is $(-\infty, \infty)$. Without any further assumptions, conclude that $\ln(x)$ has an inverse function defined on $(-\infty, \infty)$.

Exercise:

Problem:

Pretend, for the moment, that we do not know that e^x is the inverse function of $\ln(x)$, but keep in mind that $\ln(x)$ has an inverse function defined on $(-\infty, \infty)$. Call it E . Use the identity $\ln xy = \ln x + \ln y$ to deduce that $E(a + b) = E(a)E(b)$ for any real numbers a, b .

Exercise:**Problem:**

Pretend, for the moment, that we do not know that e^x is the inverse function of $\ln x$, but keep in mind that $\ln x$ has an inverse function defined on $(-\infty, \infty)$. Call it E . Show that $E'(t) = E(t)$.

Solution:

$x = E(\ln(x))$. Then, $1 = \frac{E'(\ln x)}{x}$ or $x = E'(\ln x)$. Since any number t can be written $t = \ln x$ for some x , and for such t we have $x = E(t)$, it follows that for any t , $E'(t) = E(t)$.

Exercise:**Problem:**

The sine integral, defined as $S(x) = \int_0^x \frac{\sin t}{t} dt$ is an important quantity in engineering. Although it does not have a simple closed formula, it is possible to estimate its behavior for large x . Show that for $k \geq 1$, $|S(2\pi k) - S(2\pi(k+1))| \leq \frac{1}{k(2k+1)\pi}$.
(Hint: $\sin(t + \pi) = -\sin t$)

Exercise:

Problem:

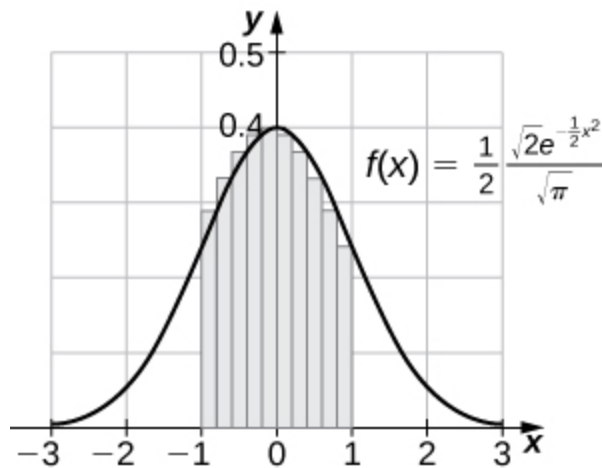
[T] The normal distribution in probability is given by

$p(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{-(x-\mu)^2/2\sigma^2}$, where σ is the standard deviation and μ is the average. The *standard normal distribution* in probability, p_s , corresponds to $\mu = 0$ and $\sigma = 1$. Compute the right endpoint

estimates R_{10} and R_{100} of $\int_{-1}^1 \frac{1}{\sqrt{2\pi}} e^{-x^2/2} dx$.

Solution:

$$R_{10} = 0.6811, R_{100} = 0.6827$$

**Exercise:****Problem:**

[T] Compute the right endpoint estimates R_{50} and R_{100} of

$$\int_{-3}^5 \frac{1}{2\sqrt{2\pi}} e^{-(x-1)^2/8} dx.$$

Integrals Resulting in Inverse Trigonometric Functions

- Integrate functions resulting in inverse trigonometric functions

In this section we focus on integrals that result in inverse trigonometric functions. We have worked with these functions before. Recall from [Functions and Graphs](#) that trigonometric functions are not one-to-one unless the domains are restricted. When working with inverses of trigonometric functions, we always need to be careful to take these restrictions into account. Also in [Derivatives](#), we developed formulas for derivatives of inverse trigonometric functions. The formulas developed there give rise directly to integration formulas involving inverse trigonometric functions.

Integrals that Result in Inverse Sine Functions

Let us begin this last section of the chapter with the three formulas. Along with these formulas, we use substitution to evaluate the integrals. We prove the formula for the inverse sine integral.

Note:

Rule: Integration Formulas Resulting in Inverse Trigonometric Functions

The following integration formulas yield inverse trigonometric functions:

1.

Equation:

$$\int \frac{du}{\sqrt{a^2 - u^2}} = \sin^{-1} \frac{u}{|a|} + C$$

2.

Equation:

$$\int \frac{du}{a^2 + u^2} = \frac{1}{a} \tan^{-1} \frac{u}{a} + C$$

3.

Equation:

$$\int \frac{du}{u\sqrt{u^2 - a^2}} = \frac{1}{|a|} \sec^{-1} \frac{u}{|a|} + C$$

Proof

Let $y = \sin^{-1} \frac{x}{a}$. Then $a \sin y = x$. Now let's use implicit differentiation.

We obtain

Equation:

$$\frac{d}{dx} (a \sin y) = \frac{d}{dx} (x)$$

$$a \cos y \frac{dy}{dx} = 1$$

$$\frac{dy}{dx} = \frac{1}{a \cos y}.$$

For $-\frac{\pi}{2} \leq y \leq \frac{\pi}{2}$, $\cos y \geq 0$. Thus, applying the Pythagorean identity

$\sin^2 y + \cos^2 y = 1$, we have $\cos y = \sqrt{1 - \sin^2 y}$. This gives

Equation:

$$\begin{aligned} \frac{1}{a \cos y} &= \frac{1}{a \sqrt{1 - \sin^2 y}} \\ &= \frac{1}{\sqrt{a^2 - a^2 \sin^2 y}} \\ &= \frac{1}{\sqrt{a^2 - x^2}}. \end{aligned}$$

Then for $-a \leq x \leq a$, and generalizing to u , we have

Equation:

$$\int \frac{1}{\sqrt{a^2 - u^2}} du = \sin^{-1} \left(\frac{u}{a} \right) + C.$$

□

Example:

Exercise:

Problem:

Evaluating a Definite Integral Using Inverse Trigonometric Functions

Evaluate the definite integral $\int_0^{\frac{1}{2}} \frac{dx}{\sqrt{1-x^2}}.$

Solution:

We can go directly to the formula for the antiderivative in the rule on integration formulas resulting in inverse trigonometric functions, and then evaluate the definite integral. We have

Equation:

$$\begin{aligned} \int_0^{\frac{1}{2}} \frac{dx}{\sqrt{1-x^2}} &= \sin^{-1} x \Big|_0^{\frac{1}{2}} \\ &= \sin^{-1} \frac{1}{2} - \sin^{-1} 0 \\ &= \frac{\pi}{6} - 0 \\ &= \frac{\pi}{6}. \end{aligned}$$

Note:

Exercise:

Problem: Find the antiderivative of $\int \frac{dx}{\sqrt{1 - 16x^2}}$.

Solution:

$$\frac{1}{4} \sin^{-1}(4x) + C$$

Hint

Substitute $u = 4x$

Example:**Exercise:****Problem:**

Finding an Antiderivative Involving an Inverse Trigonometric Function

Evaluate the integral $\int \frac{dx}{\sqrt{4 - 9x^2}}$.

Solution:

Substitute $u = 3x$. Then $du = 3dx$ and we have

Equation:

$$\int \frac{dx}{\sqrt{4 - 9x^2}} = \frac{1}{3} \int \frac{du}{\sqrt{4 - u^2}}.$$

Applying the formula with $a = 2$, we obtain

Equation:

$$\begin{aligned}
 \int \frac{dx}{\sqrt{4-9x^2}} &= \frac{1}{3} \int \frac{du}{\sqrt{4-u^2}} \\
 &= \frac{1}{3} \sin^{-1} \left(\frac{u}{2} \right) + C \\
 &= \frac{1}{3} \sin^{-1} \left(\frac{3x}{2} \right) + C.
 \end{aligned}$$

Note:

Exercise:

Problem:

Find the indefinite integral using an inverse trigonometric function and substitution for $\int \frac{dx}{\sqrt{9-x^2}}$.

Solution:

$$\sin^{-1} \left(\frac{x}{3} \right) + C$$

Hint

Use the formula in the rule on integration formulas resulting in inverse trigonometric functions.

Example:

Exercise:

Problem:

Evaluating a Definite Integral

Evaluate the definite integral $\int_0^{\sqrt{3}/2} \frac{du}{\sqrt{1-u^2}}$.

Solution:

The format of the problem matches the inverse sine formula. Thus,

Equation:

$$\begin{aligned}\int_0^{\sqrt{3}/2} \frac{du}{\sqrt{1-u^2}} &= \sin^{-1}u \Big|_0^{\sqrt{3}/2} \\ &= \left[\sin^{-1} \left(\frac{\sqrt{3}}{2} \right) \right] - [\sin^{-1}(0)] \\ &= \frac{\pi}{3}.\end{aligned}$$

Integrals Resulting in Other Inverse Trigonometric Functions

There are six inverse trigonometric functions. However, only three integration formulas are noted in the rule on integration formulas resulting in inverse trigonometric functions because the remaining three are negative versions of the ones we use. The only difference is whether the integrand is positive or negative. Rather than memorizing three more formulas, if the integrand is negative, simply factor out -1 and evaluate the integral using one of the formulas already provided. To close this section, we examine one more formula: the integral resulting in the inverse tangent function.

Example:**Exercise:****Problem:****Finding an Antiderivative Involving the Inverse Tangent Function**

Find an antiderivative of $\int \frac{1}{1+4x^2} dx$.

Solution:

Comparing this problem with the formulas stated in the rule on integration formulas resulting in inverse trigonometric functions, the integrand looks similar to the formula for $\tan^{-1}u + C$. So we use substitution, letting $u = 2x$, then $du = 2dx$ and $1/2du = dx$. Then, we have

Equation:

$$\frac{1}{2} \int \frac{1}{1 + u^2} du = \frac{1}{2} \tan^{-1}u + C = \frac{1}{2} \tan^{-1}(2x) + C.$$

Note:**Exercise:**

Problem: Use substitution to find the antiderivative $\int \frac{dx}{25 + 4x^2}$.

Solution:

$$\frac{1}{10} \tan^{-1}\left(\frac{2x}{5}\right) + C$$

Hint

Use the solving strategy from [\[link\]](#) and the rule on integration formulas resulting in inverse trigonometric functions.

Example:**Exercise:****Problem:**

Applying the Integration Formulas

Find the antiderivative of $\int \frac{1}{9 + x^2} dx$.

Solution:

Apply the formula with $a = 3$. Then,

Equation:

$$\int \frac{dx}{9 + x^2} = \frac{1}{3} \tan^{-1} \left(\frac{x}{3} \right) + C.$$

Note:

Exercise:

Problem: Find the antiderivative of $\int \frac{dx}{16 + x^2}$.

Solution:

$$\frac{1}{4} \tan^{-1} \left(\frac{x}{4} \right) + C$$

Hint

Follow the steps in [\[link\]](#).

Example:

Exercise:

Problem:

Evaluating a Definite Integral

Evaluate the definite integral $\int_{\sqrt{3}/3}^{\sqrt{3}} \frac{dx}{1+x^2}$.

Solution:

Use the formula for the inverse tangent. We have

Equation:

$$\begin{aligned}\int_{\sqrt{3}/3}^{\sqrt{3}} \frac{dx}{1+x^2} &= \tan^{-1} x \Big|_{\sqrt{3}/3}^{\sqrt{3}} \\ &= \left[\tan^{-1}(\sqrt{3}) \right] - \left[\tan^{-1}\left(\frac{\sqrt{3}}{3}\right) \right] \\ &= \frac{\pi}{6}.\end{aligned}$$

Note:

Exercise:

Problem: Evaluate the definite integral $\int_0^2 \frac{dx}{4+x^2}$.

Solution:

$$\frac{\pi}{8}$$

Hint

Follow the procedures from [\[link\]](#) to solve the problem.

Key Concepts

- Formulas for derivatives of inverse trigonometric functions developed in [Derivatives of Exponential and Logarithmic Functions](#) lead directly to integration formulas involving inverse trigonometric functions.
- Use the formulas listed in the rule on integration formulas resulting in inverse trigonometric functions to match up the correct format and make alterations as necessary to solve the problem.
- Substitution is often required to put the integrand in the correct form.

Key Equations

- **Integrals That Produce Inverse Trigonometric Functions**

$$\int \frac{du}{\sqrt{a^2 - u^2}} = \sin^{-1} \left(\frac{u}{a} \right) + C$$

$$\int \frac{du}{a^2 + u^2} = \frac{1}{a} \tan^{-1} \left(\frac{u}{a} \right) + C$$

$$\int \frac{du}{u\sqrt{u^2 - a^2}} = \frac{1}{a} \sec^{-1} \left(\frac{u}{a} \right) + C$$

In the following exercises, evaluate each integral in terms of an inverse trigonometric function.

Exercise:

Problem: $\int_0^{\sqrt{3}/2} \frac{dx}{\sqrt{1 - x^2}}$

Solution:

$$\sin^{-1} x \Big|_0^{\sqrt{3}/2} = \frac{\pi}{3}$$

Exercise:

Problem: $\int_{-1/2}^{1/2} \frac{dx}{1 - x^2}$

Exercise:

Problem: $\int_{\sqrt{3}}^1 \frac{dx}{1-x^2}$

Solution:

$$\tan^{-1}x \Big|_{\sqrt{3}}^1 = -\frac{\pi}{12}$$

Exercise:

Problem: $\int_{1/\sqrt{3}}^{\sqrt{3}} \frac{dx}{1+x^2}$

Exercise:

Problem: $\int_1^{\sqrt{2}} \frac{dx}{|x|\sqrt{x^2-1}}$

Solution:

$$\sec^{-1}x \Big|_1^{\sqrt{2}} = \frac{\pi}{4}$$

Exercise:

Problem: $\int_1^{2/\sqrt{3}} \frac{dx}{|x|\sqrt{x^2-1}}$

In the following exercises, find each indefinite integral, using appropriate substitutions.

Exercise:

Problem: $\int \frac{dx}{\sqrt{9-x^2}}$

Solution:

$$\sin^{-1}\left(\frac{x}{3}\right) + C$$

Exercise:

Problem: $\int \frac{dx}{\sqrt{1-16x^2}}$

Exercise:

Problem: $\int \frac{dx}{9+x^2}$

Solution:

$$\frac{1}{3} \tan^{-1}\left(\frac{x}{3}\right) + C$$

Exercise:

Problem: $\int \frac{dx}{25+16x^2}$

Exercise:

Problem: $\int \frac{dx}{|x|\sqrt{x^2-9}}$

Solution:

$$\frac{1}{3} \sec^{-1}\left(\frac{x}{3}\right) + C$$

Exercise:

Problem: $\int \frac{dx}{|x|\sqrt{4x^2-16}}$

Exercise:**Problem:**

Explain the relationship $-\cos^{-1}t + C = \int \frac{dt}{\sqrt{1-t^2}} = \sin^{-1}t + C$.
Is it true, in general, that $\cos^{-1}t = -\sin^{-1}t$?

Solution:

$\cos\left(\frac{\pi}{2} - \theta\right) = \sin\theta$. So, $\sin^{-1}t = \frac{\pi}{2} - \cos^{-1}t$. They differ by a constant.

Exercise:**Problem:**

Explain the relationship
 $\sec^{-1}t + C = \int \frac{dt}{|t|\sqrt{t^2-1}} = -\csc^{-1}t + C$. Is it true, in general,
that $\sec^{-1}t = -\csc^{-1}t$?

Exercise:**Problem:**

Explain what is wrong with the following integral: $\int_1^2 \frac{dt}{\sqrt{1-t^2}}$.

Solution:

$\sqrt{1-t^2}$ is not defined as a real number when $t > 1$.

Exercise:**Problem:**

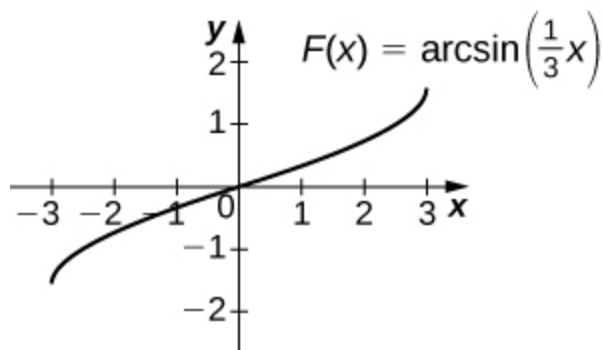
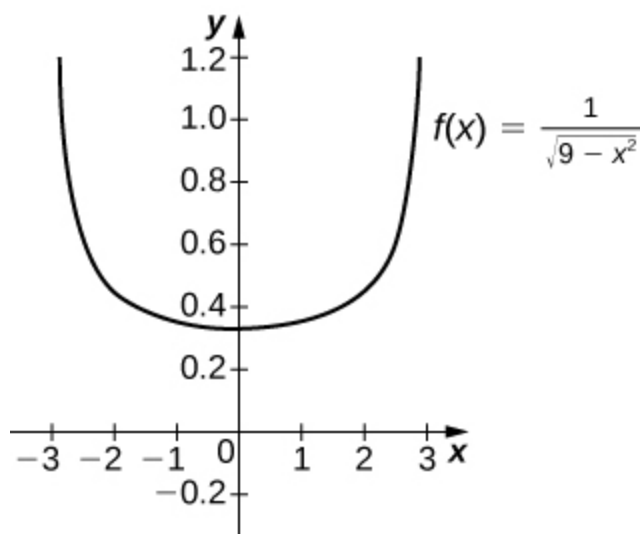
Explain what is wrong with the following integral: $\int_{-1}^1 \frac{dt}{|t|\sqrt{t^2-1}}$.

In the following exercises, solve for the antiderivative $\int f$ of f with $C = 0$, then use a calculator to graph f and the antiderivative over the given interval $[a, b]$. Identify a value of C such that adding C to the antiderivative recovers the definite integral $F(x) = \int_a^x f(t)dt$.

Exercise:

Problem: [T] $\int \frac{1}{\sqrt{9-x^2}} dx$ over $[-3, 3]$

Solution:



The antiderivative is $\sin^{-1}\left(\frac{x}{3}\right) + C$. Taking $C = \frac{\pi}{2}$ recovers the definite integral.

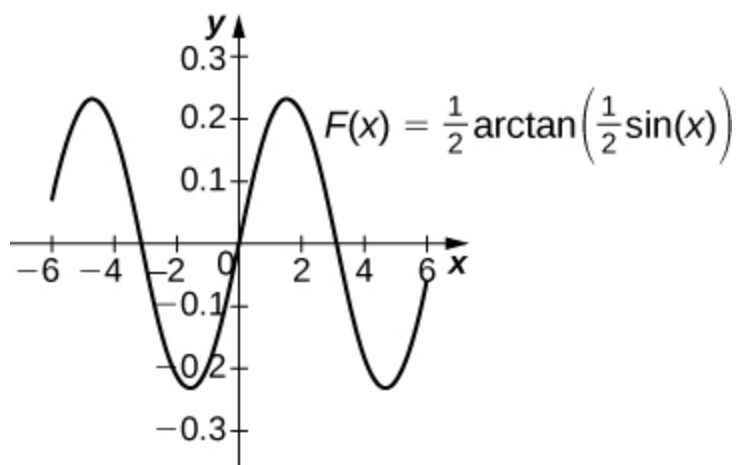
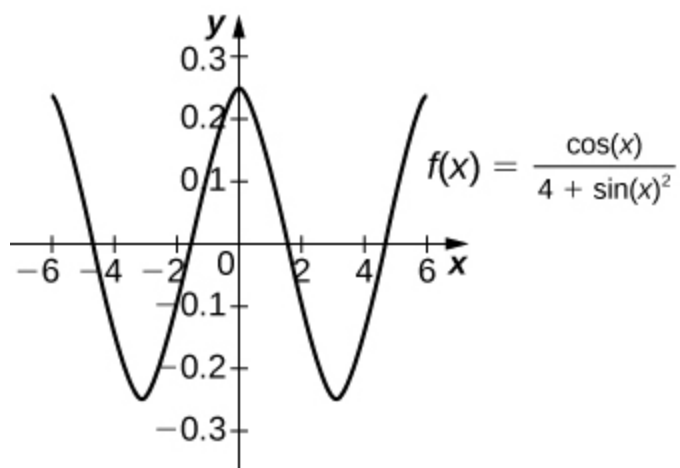
Exercise:

Problem: [T] $\int \frac{9}{9 + x^2} dx$ over $[-6, 6]$

Exercise:

Problem: [T] $\int \frac{\cos x}{4 + \sin^2 x} dx$ over $[-6, 6]$

Solution:



The antiderivative is $\frac{1}{2} \tan^{-1} \left(\frac{\sin x}{2} \right) + C$. Taking $C = \frac{1}{2} \tan^{-1} \left(\frac{\sin(6)}{2} \right)$ recovers the definite integral.

Exercise:

Problem: [T] $\int \frac{e^x}{1 + e^{2x}} dx$ over $[-6, 6]$

In the following exercises, compute the antiderivative using appropriate substitutions.

Exercise:

Problem: $\int \frac{\sin^{-1} t dt}{\sqrt{1-t^2}}$

Solution:

$$\frac{1}{2} (\sin^{-1} t)^2 + C$$

Exercise:

Problem: $\int \frac{dt}{\sin^{-1} t \sqrt{1-t^2}}$

Exercise:

Problem: $\int \frac{\tan^{-1} (2t)}{1+4t^2} dt$

Solution:

$$\frac{1}{4} (\tan^{-1} (2t))^2$$

Exercise:

Problem: $\int \frac{t \tan^{-1} (t^2)}{1+t^4} dt$

Exercise:

Problem: $\int \frac{\sec^{-1} \left(\frac{t}{2} \right)}{|t| \sqrt{t^2 - 4}} dt$

Solution:

$$\frac{1}{4} \left(\sec^{-1} \left(\frac{t}{2} \right)^2 \right) + C$$

Exercise:

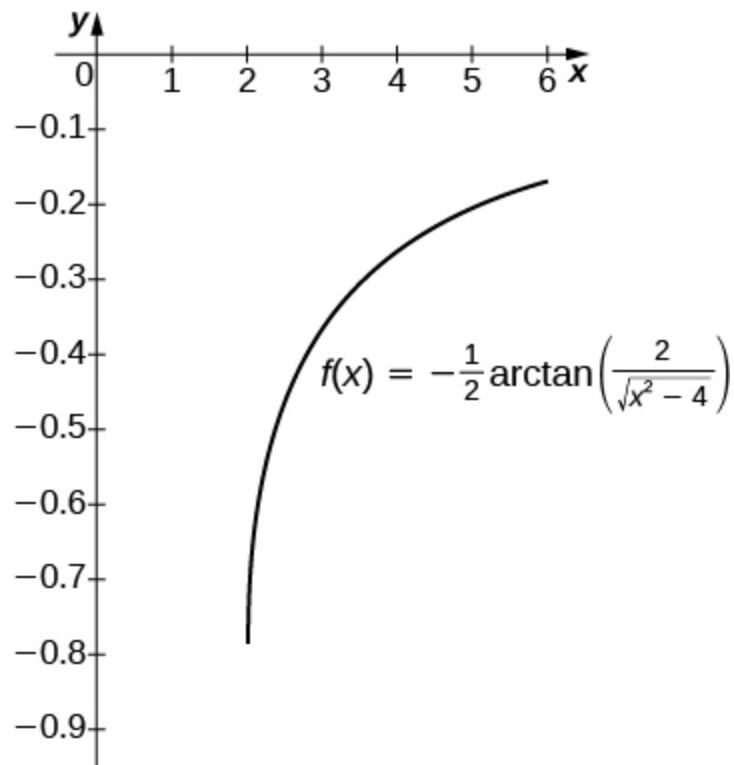
Problem: $\int \frac{t \sec^{-1}(t^2)}{t^2 \sqrt{t^4 - 1}} dt$

In the following exercises, use a calculator to graph the antiderivative $\int f$ with $C = 0$ over the given interval $[a, b]$. Approximate a value of C , if possible, such that adding C to the antiderivative gives the same value as the definite integral $F(x) = \int_a^x f(t) dt$.

Exercise:

Problem: [T] $\int \frac{1}{x \sqrt{x^2 - 4}} dx$ over $[2, 6]$

Solution:



The antiderivative is $\frac{1}{2} \sec^{-1} \left(\frac{x}{2} \right) + C$. Taking $C = 0$ recovers the definite integral over $[2, 6]$.

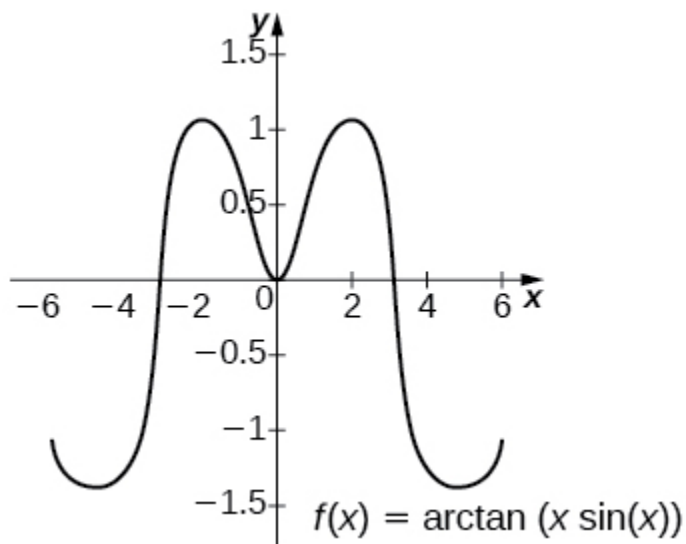
Exercise:

Problem: [T] $\int \frac{1}{(2x+2)\sqrt{x}} dx$ over $[0, 6]$

Exercise:

Problem: [T] $\int \frac{(\sin x + x \cos x)}{1 + x^2 \sin^2 x} dx$ over $[-6, 6]$

Solution:



The general antiderivative is $\tan^{-1}(x \sin x) + C$. Taking $C = -\tan^{-1}(6 \sin(6))$ recovers the definite integral.

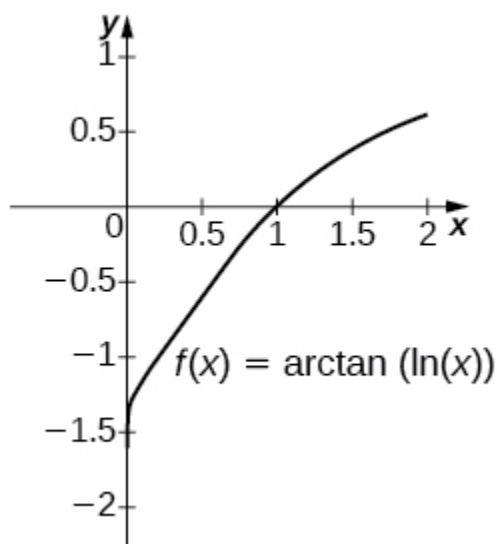
Exercise:

Problem: [T] $\int \frac{2e^{-2x}}{\sqrt{1 - e^{-4x}}} dx$ over $[0, 2]$

Exercise:

Problem: [T] $\int \frac{1}{x + x \ln^2 x} dx$ over $[0, 2]$

Solution:



The general antiderivative is $\tan^{-1}(\ln x) + C$. Taking $C = \frac{\pi}{2} = \tan^{-1}\infty$ recovers the definite integral.

Exercise:

Problem: [T] $\int \frac{\sin^{-1} x}{\sqrt{1-x^2}}$ over $[-1, 1]$

In the following exercises, compute each integral using appropriate substitutions.

Exercise:

Problem: $\int \frac{e^t}{\sqrt{1-e^{2t}}} dt$

Solution:

$$\sin^{-1}(e^t) + C$$

Exercise:

Problem: $\int \frac{e^t}{1 + e^{2t}} dt$

Exercise:

Problem: $\int \frac{dt}{t\sqrt{1 - \ln^2 t}}$

Solution:

$$\sin^{-1}(\ln t) + C$$

Exercise:

Problem: $\int \frac{dt}{t(1 + \ln^2 t)}$

Exercise:

Problem: $\int \frac{\cos^{-1}(2t)}{\sqrt{1 - 4t^2}} dt$

Solution:

$$-\frac{1}{2}(\cos^{-1}(2t))^2 + C$$

Exercise:

Problem: $\int \frac{e^t \cos^{-1}(e^t)}{\sqrt{1 - e^{2t}}} dt$

In the following exercises, compute each definite integral.

Exercise:

Problem: $\int_0^{1/2} \frac{\tan(\sin^{-1}t)}{\sqrt{1-t^2}} dt$

Solution:

$$\frac{1}{2} \ln\left(\frac{4}{3}\right)$$

Exercise:

Problem: $\int_{1/4}^{1/2} \frac{\tan(\cos^{-1}t)}{\sqrt{1-t^2}} dt$

Exercise:

Problem: $\int_0^{1/2} \frac{\sin(\tan^{-1}t)}{1+t^2} dt$

Solution:

$$1 - \frac{2}{\sqrt{5}}$$

Exercise:

Problem: $\int_0^{1/2} \frac{\cos(\tan^{-1}t)}{1+t^2} dt$

Exercise:

Problem:

For $A > 0$, compute $I(A) = \int_{-A}^A \frac{dt}{1+t^2}$ and evaluate $\lim_{A \rightarrow \infty} I(A)$, the area under the graph of $\frac{1}{1+t^2}$ on $[-\infty, \infty]$.

Solution:

$$2\tan^{-1}(A) \rightarrow \pi \text{ as } A \rightarrow \infty$$

Exercise:

Problem:

For $1 < B < \infty$, compute $I(B) = \int_1^B \frac{dt}{t\sqrt{t^2-1}}$ and evaluate $\lim_{B \rightarrow \infty} I(B)$, the area under the graph of $\frac{1}{t\sqrt{t^2-1}}$ over $[1, \infty)$.

Exercise:

Problem:

Use the substitution $u = \sqrt{2} \cot x$ and the identity $1 + \cot^2 x = \csc^2 x$ to evaluate $\int \frac{dx}{1 + \cos^2 x}$. (*Hint: Multiply the top and bottom of the integrand by $\csc^2 x$.*)

Solution:

Using the hint, one has $\int \frac{\csc^2 x}{\csc^2 x + \cot^2 x} dx = \int \frac{\csc^2 x}{1 + 2\cot^2 x} dx$.

Set $u = \sqrt{2}\cot x$. Then, $du = -\sqrt{2}\csc^2 x$ and the integral is $-\frac{1}{\sqrt{2}} \int \frac{du}{1 + u^2} = -\frac{1}{\sqrt{2}} \tan^{-1} u + C = \frac{1}{\sqrt{2}} \tan^{-1} (\sqrt{2}\cot x) + C$.

If one uses the identity $\tan^{-1} s + \tan^{-1} \left(\frac{1}{s}\right) = \frac{\pi}{2}$, then this can also be written $\frac{1}{\sqrt{2}} \tan^{-1} \left(\frac{\tan x}{\sqrt{2}}\right) + C$.

Exercise:

Problem:

[T] Approximate the points at which the graphs of $f(x) = 2x^2 - 1$ and $g(x) = (1 + 4x^2)^{-3/2}$ intersect to four decimal places, and approximate the area between their graphs to three decimal places.

Exercise:

Problem:

47. [T] Approximate the points at which the graphs of $f(x) = x^2 - 1$ and $g(x) = (x^2 + 1)^{\frac{1}{2}}$ intersect to four decimal places, and approximate the area between their graphs to three decimal places.

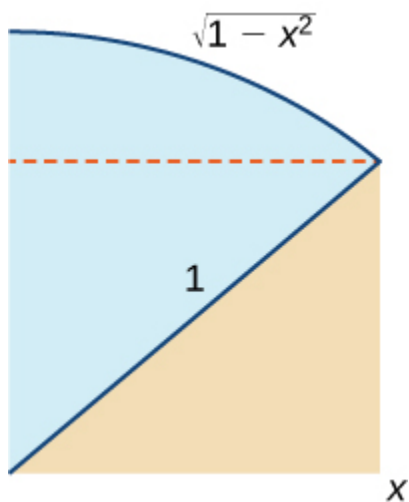
Solution:

$x \approx \pm 1.7321$. The left endpoint estimate with $N = 100$ is 4.781 and these decimals persist for $N = 500$.

Exercise:**Problem:**

Use the following graph to prove that

$$\int_0^x \sqrt{1-t^2} dt = \frac{1}{2}x\sqrt{1-x^2} + \frac{1}{2}\sin^{-1}x.$$

**Chapter Review Exercises**

True or False. Justify your answer with a proof or a counterexample. Assume all functions f and g are continuous over their domains.

Exercise:

Problem:

If $f(x) > 0$, $f'(x) > 0$ for all x , then the right-hand rule underestimates the integral $\int_a^b f(x)$. Use a graph to justify your answer.

Solution:

False

Exercise:

Problem: $\int_a^b f(x)^2 dx = \int_a^b f(x) dx \int_a^b f(x) dx$

Exercise:

Problem:

If $f(x) \leq g(x)$ for all $x \in [a, b]$, then $\int_a^b f(x) \leq \int_a^b g(x)$.

Solution:

True

Exercise:

Problem: All continuous functions have an antiderivative.

Evaluate the Riemann sums L_4 and R_4 for the following functions over the specified interval. Compare your answer with the exact answer, when possible, or use a calculator to determine the answer.

Exercise:

Problem: $y = 3x^2 - 2x + 1$ over $[-1, 1]$

Solution:

$$L_4 = 5.25, R_4 = 3.25, \text{ exact answer: } 4$$

Exercise:

Problem: $y = \ln(x^2 + 1)$ over $[0, e]$

Exercise:

Problem: $y = x^2 \sin x$ over $[0, \pi]$

Solution:

$$L_4 = 5.364, R_4 = 5.364, \text{ exact answer: } 5.870$$

Exercise:

Problem: $y = \sqrt{x} + \frac{1}{x}$ over $[1, 4]$

Evaluate the following integrals.

Exercise:

Problem: $\int_{-1}^1 (x^3 - 2x^2 + 4x) dx$

Solution:

$$-\frac{4}{3}$$

Exercise:

Problem: $\int_0^4 \frac{3t}{\sqrt{1+6t^2}} dt$

Exercise:

Problem: $\int_{\pi/3}^{\pi/2} 2 \sec(2\theta) \tan(2\theta) d\theta$

Solution:

1

Exercise:

Problem: $\int_0^{\pi/4} e^{\cos^2 x} \sin x \cos x dx$

Find the antiderivative.

Exercise:

Problem: $\int \frac{dx}{(x+4)^3}$

Solution:

$$-\frac{1}{2(x+4)^2} + C$$

Exercise:

Problem: $\int x \ln(x^2) dx$

Exercise:

Problem: $\int \frac{4x^2}{\sqrt{1-x^6}} dx$

Solution:

$$\frac{4}{3} \sin^{-1}(x^3) + C$$

Exercise:

Problem: $\int \frac{e^{2x}}{1+e^{4x}} dx$

Find the derivative.

Exercise:

Problem: $\frac{d}{dt} \int_0^t \frac{\sin x}{\sqrt{1+x^2}} dx$

Solution:

$$\frac{\sin t}{\sqrt{1+t^2}}$$

Exercise:

Problem: $\frac{d}{dx} \int_1^{x^3} \sqrt{4-t^2} dt$

Exercise:

Problem: $\frac{d}{dx} \int_1^{\ln(x)} (4t + e^t) dt$

Solution:

$$4 \frac{\ln x}{x} + 1$$

Exercise:

Problem: $\frac{d}{dx} \int_0^{\cos x} e^{t^2} dt$

The following problems consider the historic average cost per gigabyte of RAM on a computer.

Year	5-Year Change (\$)
1980	0
1985	-5,468,750
1990	-755,495
1995	-73,005
2000	-29,768
2005	-918
2010	-177

Exercise:

Problem:

If the average cost per gigabyte of RAM in 2010 is \$12, find the average cost per gigabyte of RAM in 1980.

Solution:

\$6,328,113

Exercise:**Problem:**

The average cost per gigabyte of RAM can be approximated by the function $C(t) = 8,500,000(0.65)^t$, where t is measured in years since 1980, and C is cost in US\$. Find the average cost per gigabyte of RAM for 1980 to 2010.

Exercise:

Problem: Find the average cost of 1GB RAM for 2005 to 2010.

Solution:

\$73.36

Exercise:**Problem:**

The velocity of a bullet from a rifle can be approximated by $v(t) = 6400t^2 - 6505t + 2686$, where t is seconds after the shot and v is the velocity measured in feet per second. This equation only models the velocity for the first half-second after the shot: $0 \leq t \leq 0.5$. What is the total distance the bullet travels in 0.5 sec?

Exercise:**Problem:**

What is the average velocity of the bullet for the first half-second?

Solution:

$\frac{19117}{12}$ ft/sec, or 1593 ft/sec

Determining Volumes by Slicing

- Determine the volume of a solid by integrating a cross-section (the slicing method).
- Find the volume of a solid of revolution using the disk method.
- Find the volume of a solid of revolution with a cavity using the washer method.

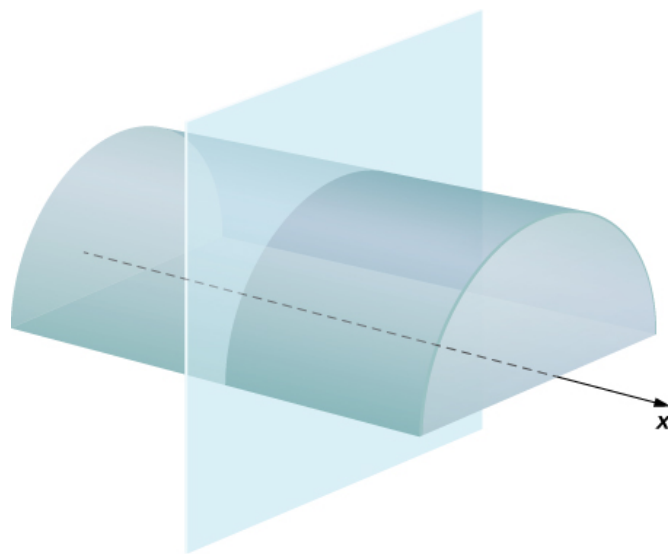
In the preceding section, we used definite integrals to find the area between two curves. In this section, we use definite integrals to find volumes of three-dimensional solids. We consider three approaches—slicing, disks, and washers—for finding these volumes, depending on the characteristics of the solid.

Volume and the Slicing Method

Just as area is the numerical measure of a two-dimensional region, volume is the numerical measure of a three-dimensional solid. Most of us have computed volumes of solids by using basic geometric formulas. The volume of a rectangular solid, for example, can be computed by multiplying length, width, and height: $V = lwh$. The formulas for the volume of a sphere ($V = \frac{4}{3}\pi r^3$), a cone ($V = \frac{1}{3}\pi r^2 h$), and a pyramid ($V = \frac{1}{3}Ah$) have also been introduced. Although some of these formulas were derived using geometry alone, all these formulas can be obtained by using integration.

We can also calculate the volume of a cylinder. Although most of us think of a cylinder as having a circular base, such as a soup can or a metal rod, in mathematics the word *cylinder* has a more general meaning. To discuss cylinders in this more general context, we first need to define some vocabulary.

We define the **cross-section** of a solid to be the intersection of a plane with the solid. A *cylinder* is defined as any solid that can be generated by translating a plane region along a line perpendicular to the region, called the *axis* of the cylinder. Thus, all cross-sections perpendicular to the axis of a cylinder are identical. The solid shown in [\[link\]](#) is an example of a cylinder with a noncircular base. To calculate the volume of a cylinder, then, we simply multiply the area of the cross-section by the height of the cylinder: $V = A \cdot h$. In the case of a right circular cylinder (soup can), this becomes $V = \pi r^2 h$.



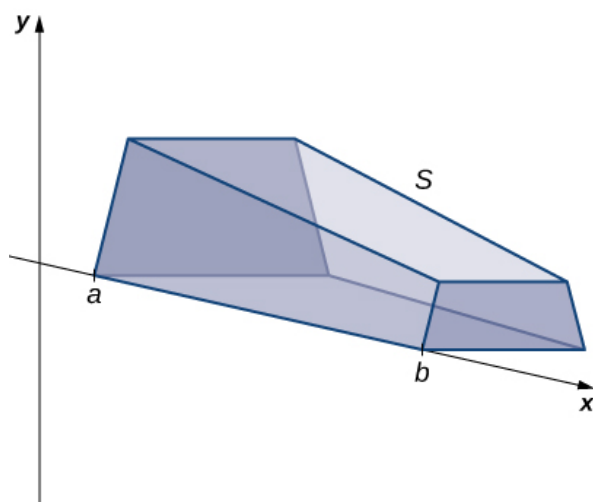
Three-dimensional cylinder



Two-dimensional cross section

Each cross-section of a particular cylinder is identical to the others.

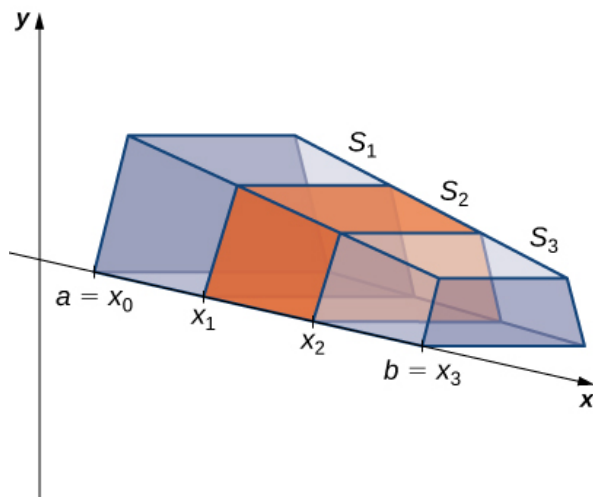
If a solid does not have a constant cross-section (and it is not one of the other basic solids), we may not have a formula for its volume. In this case, we can use a definite integral to calculate the volume of the solid. We do this by slicing the solid into pieces, estimating the volume of each slice, and then adding those estimated volumes together. The slices should all be parallel to one another, and when we put all the slices together, we should get the whole solid. Consider, for example, the solid S shown in [\[link\]](#), extending along the x -axis.



A solid with a varying cross-section.

We want to divide S into slices perpendicular to the x -axis. As we see later in the chapter, there may be times when we want to slice the solid in some other direction—say, with slices perpendicular to the y -axis. The decision of which way to slice the solid is very important. If we make the wrong choice, the computations can get quite messy. Later in the chapter, we examine some of these situations in detail and look at how to decide which way to slice the solid. For the purposes of this section, however, we use slices perpendicular to the x -axis.

Because the cross-sectional area is not constant, we let $A(x)$ represent the area of the cross-section at point x . Now let $P = \{x_0, x_1, \dots, x_n\}$ be a regular partition of $[a, b]$, and for $i = 1, 2, \dots, n$, let S_i represent the slice of S stretching from x_{i-1} to x_i . The following figure shows the sliced solid with $n = 3$.



The solid S has been divided into three slices perpendicular to the x -axis.

Finally, for $i = 1, 2, \dots, n$, let x_i^* be an arbitrary point in $[x_{i-1}, x_i]$. Then the volume of slice S_i can be estimated by $V(S_i) \approx A(x_i^*)\Delta x$. Adding these approximations together, we see the volume of the entire solid S can be approximated by

Equation:

$$V(S) \approx \sum_{i=1}^n A(x_i^*)\Delta x.$$

By now, we can recognize this as a Riemann sum, and our next step is to take the limit as $n \rightarrow \infty$. Then we have

Equation:

$$V(S) = \lim_{n \rightarrow \infty} \sum_{i=1}^n A(x_i^*)\Delta x = \int_a^b A(x)dx.$$

The technique we have just described is called the **slicing method**. To apply it, we use the following strategy.

Note:

Problem-Solving Strategy: Finding Volumes by the Slicing Method

1. Examine the solid and determine the shape of a cross-section of the solid. It is often helpful to draw a picture if one is not provided.
2. Determine a formula for the area of the cross-section.
3. Integrate the area formula over the appropriate interval to get the volume.

Recall that in this section, we assume the slices are perpendicular to the x -axis. Therefore, the area formula is in terms of x and the limits of integration lie on the x -axis. However, the problem-solving strategy shown here is valid regardless of how we choose to slice the solid.

Example:

Exercise:

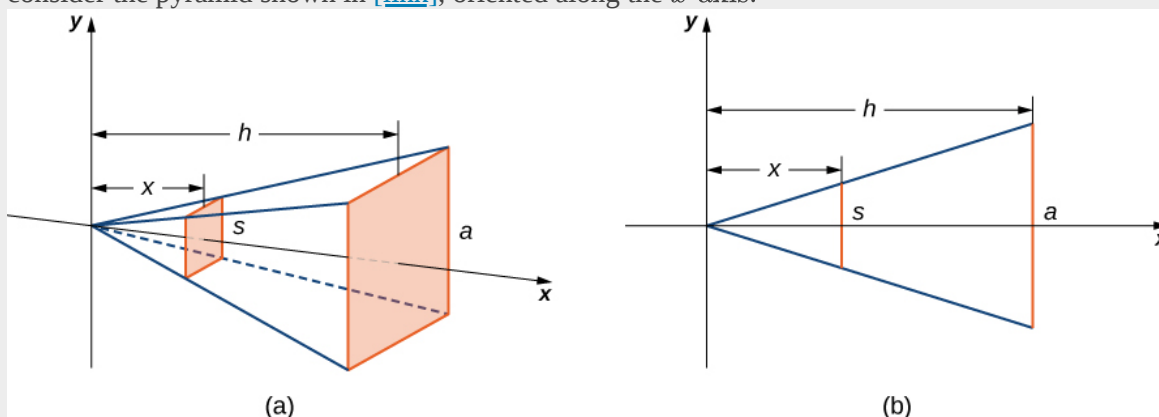
Problem:

Deriving the Formula for the Volume of a Pyramid

We know from geometry that the formula for the volume of a pyramid is $V = \frac{1}{3}Ah$. If the pyramid has a square base, this becomes $V = \frac{1}{3}a^2h$, where a denotes the length of one side of the base. We are going to use the slicing method to derive this formula.

Solution:

We want to apply the slicing method to a pyramid with a square base. To set up the integral, consider the pyramid shown in [\[link\]](#), oriented along the x -axis.



(a) A pyramid with a square base is oriented along the x -axis. (b) A two-dimensional view of the pyramid is seen from the side.

We first want to determine the shape of a cross-section of the pyramid. We know the base is a square, so the cross-sections are squares as well (step 1). Now we want to determine a formula for the area of one of these cross-sectional squares. Looking at [\[link\]](#)(b), and using a proportion, since these are similar triangles, we have

Equation:

$$\frac{s}{a} = \frac{x}{h} \text{ or } s = \frac{ax}{h}.$$

Therefore, the area of one of the cross-sectional squares is

Equation:

$$A(x) = s^2 = \left(\frac{ax}{h}\right)^2 \text{ (step 2).}$$

Then we find the volume of the pyramid by integrating from 0 to h (step 3):

Equation:

$$\begin{aligned} V &= \int_0^h A(x) dx \\ &= \int_0^h \left(\frac{ax}{h}\right)^2 dx = \frac{a^2}{h^2} \int_0^h x^2 dx \\ &= \left[\frac{a^2}{h^2} \left(\frac{1}{3} x^3\right) \right] \Big|_0^h = \frac{1}{3} a^2 h. \end{aligned}$$

This is the formula we were looking for.

Note:

Exercise:

Problem:

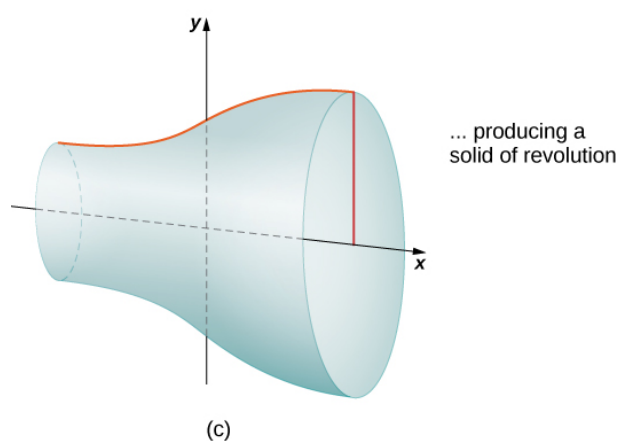
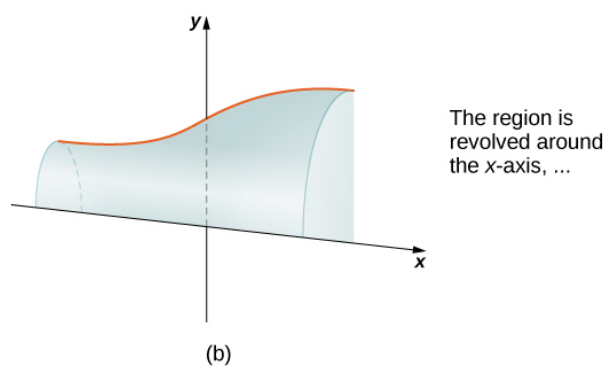
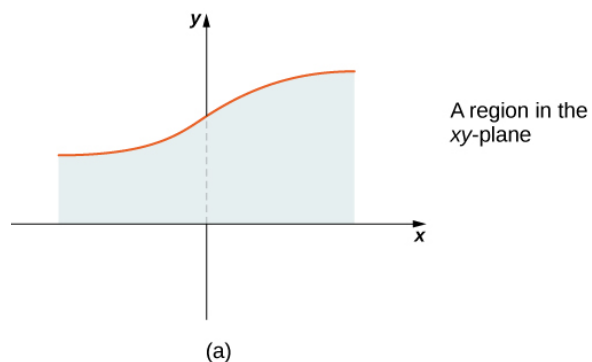
Use the slicing method to derive the formula $V = \frac{1}{3} \pi r^2 h$ for the volume of a circular cone.

Hint

Use similar triangles, as in [\[link\]](#).

Solids of Revolution

If a region in a plane is revolved around a line in that plane, the resulting solid is called a **solid of revolution**, as shown in the following figure.



(a) This is the region that is revolved around the x -axis. (b) As the region begins to revolve around the axis, it sweeps out a solid of revolution. (c) This is the solid that results when the revolution is complete.

Solids of revolution are common in mechanical applications, such as machine parts produced by a lathe. We spend the rest of this section looking at solids of this type. The next example uses the slicing method to calculate the volume of a solid of revolution.

Note:

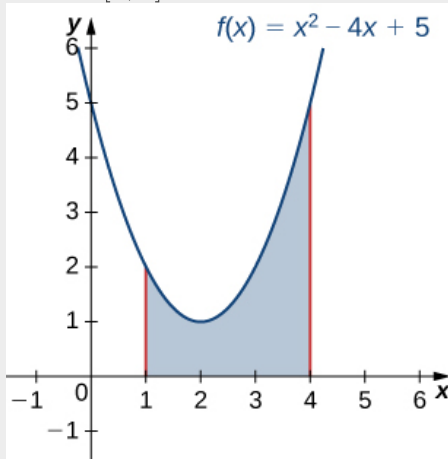
Use an online [integral calculator](#) to learn more.

Example:**Exercise:****Problem:****Using the Slicing Method to find the Volume of a Solid of Revolution**

Use the slicing method to find the volume of the solid of revolution bounded by the graphs of $f(x) = x^2 - 4x + 5$, $x = 1$, and $x = 4$, and rotated about the x -axis.

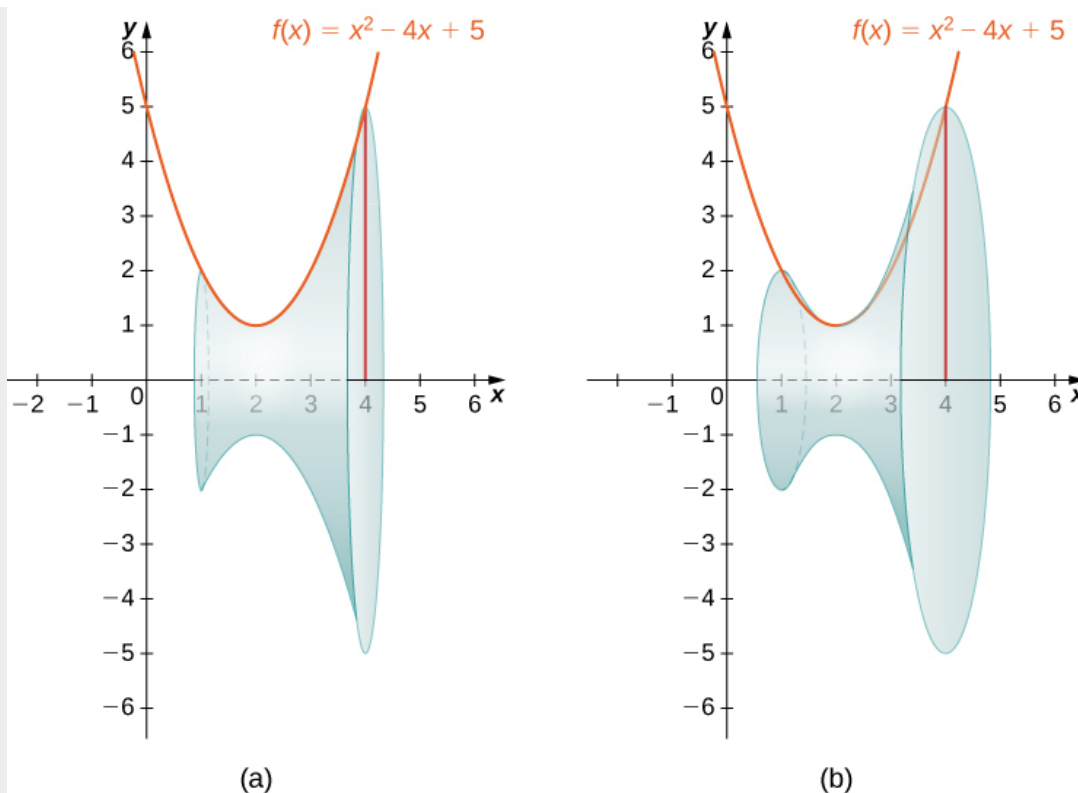
Solution:

Using the problem-solving strategy, we first sketch the graph of the quadratic function over the interval $[1, 4]$ as shown in the following figure.



A region used to produce a solid of revolution.

Next, revolve the region around the x -axis, as shown in the following figure.



Two views, (a) and (b), of the solid of revolution produced by revolving the region in [\[link\]](#) about the x -axis.

Since the solid was formed by revolving the region around the x -axis, the cross-sections are circles (step 1). The area of the cross-section, then, is the area of a circle, and the radius of the circle is given by $f(x)$. Use the formula for the area of the circle:

Equation:

$$A(x) = \pi r^2 = \pi [f(x)]^2 = \pi (x^2 - 4x + 5)^2 \text{ (step 2).}$$

The volume, then, is (step 3)

Equation:

$$\begin{aligned} V &= \int_a^b A(x) dx \\ &= \int_1^4 \pi (x^2 - 4x + 5)^2 dx = \pi \int_1^4 (x^4 - 8x^3 + 26x^2 - 40x + 25) dx \\ &= \pi \left(\frac{x^5}{5} - 2x^4 + \frac{26x^3}{3} - 20x^2 + 25x \right) \Big|_1^4 = \frac{78}{5} \pi. \end{aligned}$$

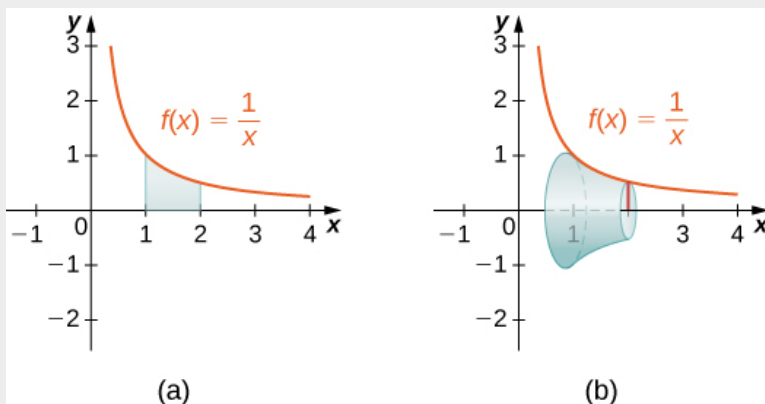
The volume is $78\pi/5$.

Note:

Exercise:

Problem:

Use the method of slicing to find the volume of the solid of revolution formed by revolving the region between the graph of the function $f(x) = 1/x$ and the x -axis over the interval $[1, 2]$ around the x -axis. See the following figure.



Solution:

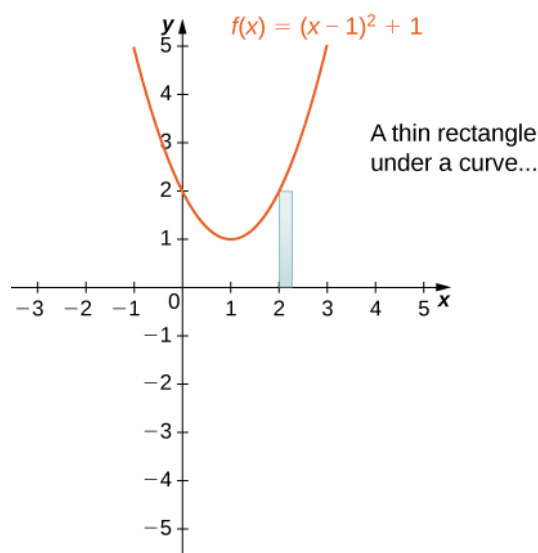
$$\frac{\pi}{2}$$

Hint

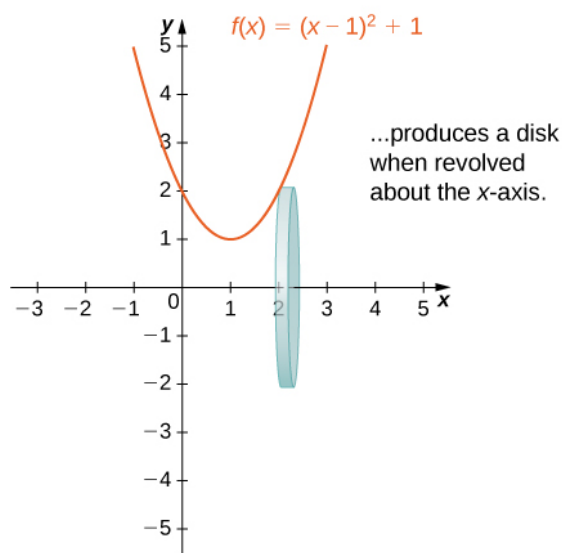
Use the problem-solving strategy presented earlier and follow [\[link\]](#) to help with step 2.

The Disk Method

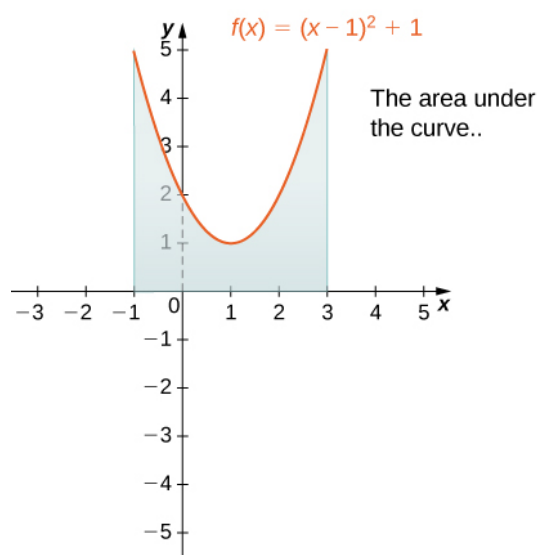
When we use the slicing method with solids of revolution, it is often called the **disk method** because, for solids of revolution, the slices used to over approximate the volume of the solid are disks. To see this, consider the solid of revolution generated by revolving the region between the graph of the function $f(x) = (x - 1)^2 + 1$ and the x -axis over the interval $[-1, 3]$ around the x -axis. The graph of the function and a representative disk are shown in [\[link\]](#)(a) and (b). The region of revolution and the resulting solid are shown in [\[link\]](#)(c) and (d).



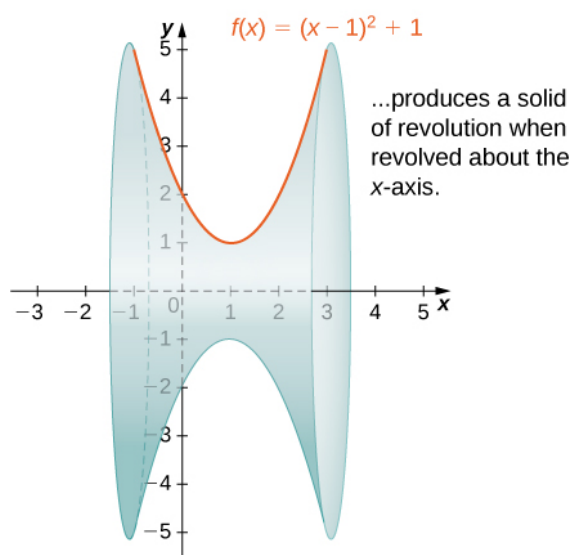
(a)



(b)



(c)



(d)

(a) A thin rectangle for approximating the area under a curve. (b) A representative disk formed by revolving the rectangle about the x -axis. (c) The region under the curve is revolved about the x -axis, resulting in (d) the solid of revolution.

We already used the formal Riemann sum development of the volume formula when we developed the slicing method. We know that

Equation:

$$V = \int_a^b A(x) dx.$$

The only difference with the disk method is that we know the formula for the cross-sectional area ahead of time; it is the area of a circle. This gives the following rule.

Note:

Rule: The Disk Method

Let $f(x)$ be continuous and nonnegative. Define R as the region bounded above by the graph of $f(x)$, below by the x -axis, on the left by the line $x = a$, and on the right by the line $x = b$. Then, the volume of the solid of revolution formed by revolving R around the x -axis is given by

Equation:

$$V = \int_a^b \pi [f(x)]^2 dx.$$

The volume of the solid we have been studying ([link](#)) is given by

Equation:

$$\begin{aligned} V &= \int_a^b \pi [f(x)]^2 dx \\ &= \int_{-1}^3 \pi [(x-1)^2 + 1]^2 dx = \pi \int_{-1}^3 [(x-1)^4 + 2(x-1)^2 + 1] dx \\ &= \pi \left[\frac{1}{5}(x-1)^5 + \frac{2}{3}(x-1)^3 + x \right] \Big|_{-1}^3 = \pi \left[\left(\frac{32}{5} + \frac{16}{3} + 3 \right) - \left(-\frac{32}{5} - \frac{16}{3} - 1 \right) \right] = \frac{412\pi}{15} \text{ units}^3. \end{aligned}$$

Let's look at some examples.

Example:

Exercise:

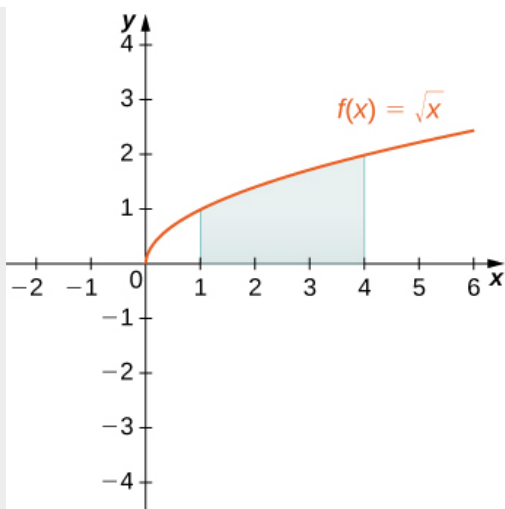
Problem:

Using the Disk Method to Find the Volume of a Solid of Revolution 1

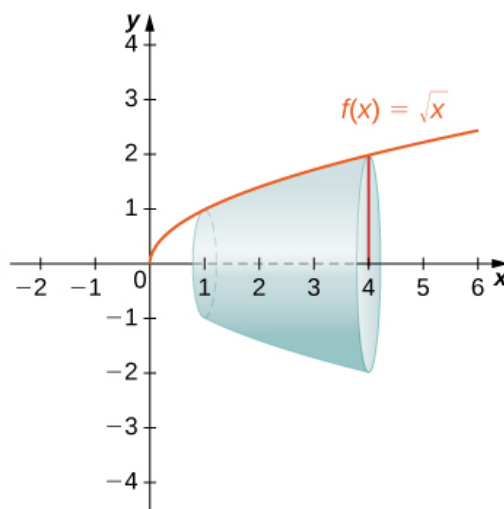
Use the disk method to find the volume of the solid of revolution generated by rotating the region between the graph of $f(x) = \sqrt{x}$ and the x -axis over the interval $[1, 4]$ around the x -axis.

Solution:

The graphs of the function and the solid of revolution are shown in the following figure.



(a)



(b)

(a) The function $f(x) = \sqrt{x}$ over the interval $[1, 4]$. (b) The solid of revolution obtained by revolving the region under the graph of $f(x)$ about the x -axis.

We have

Equation:

$$\begin{aligned} V &= \int_a^b \pi [f(x)]^2 dx \\ &= \int_1^4 \pi [\sqrt{x}]^2 dx = \pi \int_1^4 x dx \\ &= \frac{\pi}{2} x^2 \Big|_1^4 = \frac{15\pi}{2}. \end{aligned}$$

The volume is $(15\pi)/2$ units³.

Note:

Exercise:

Problem:

Use the disk method to find the volume of the solid of revolution generated by rotating the region between the graph of $f(x) = \sqrt{4-x}$ and the x -axis over the interval $[0, 4]$ around the x -axis.

Solution:

8π units³

Hint

Use the procedure from [\[link\]](#).

So far, our examples have all concerned regions revolved around the x -axis, but we can generate a solid of revolution by revolving a plane region around any horizontal or vertical line. In the next example, we look at a solid of revolution that has been generated by revolving a region around the y -axis. The mechanics of the disk method are nearly the same as when the x -axis is the axis of revolution, but we express the function in terms of y and we integrate with respect to y as well. This is summarized in the following rule.

Note:

Rule: The Disk Method for Solids of Revolution around the y -axis

Let $g(y)$ be continuous and nonnegative. Define Q as the region bounded on the right by the graph of $g(y)$, on the left by the y -axis, below by the line $y = c$, and above by the line $y = d$. Then, the volume of the solid of revolution formed by revolving Q around the y -axis is given by

Equation:

$$V = \int_c^d \pi[g(y)]^2 dy.$$

The next example shows how this rule works in practice.

Example:

Exercise:

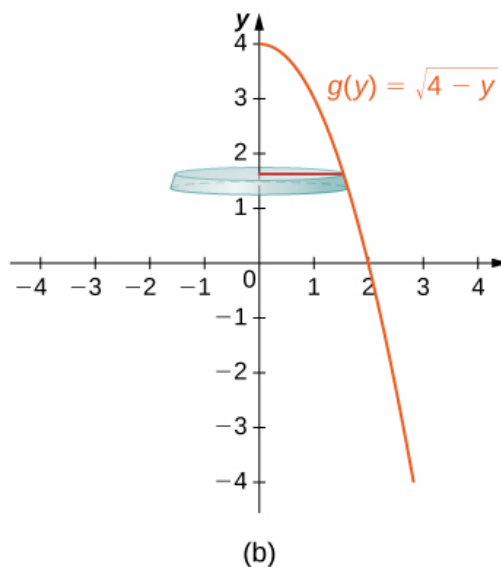
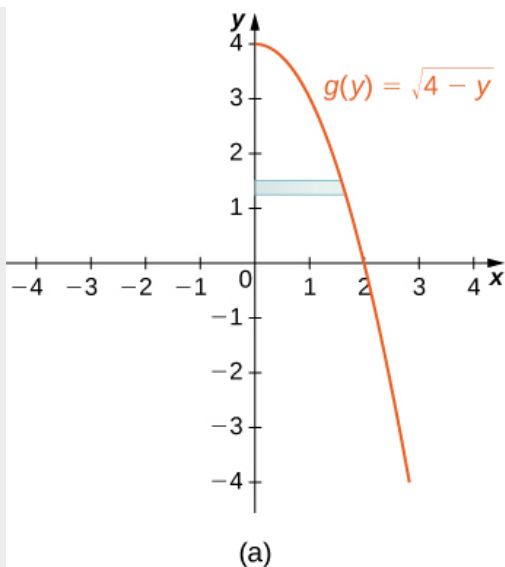
Problem:

Using the Disk Method to Find the Volume of a Solid of Revolution 2

Let R be the region bounded by the graph of $g(y) = \sqrt{4 - y}$ and the y -axis over the y -axis interval $[0, 4]$. Use the disk method to find the volume of the solid of revolution generated by rotating R around the y -axis.

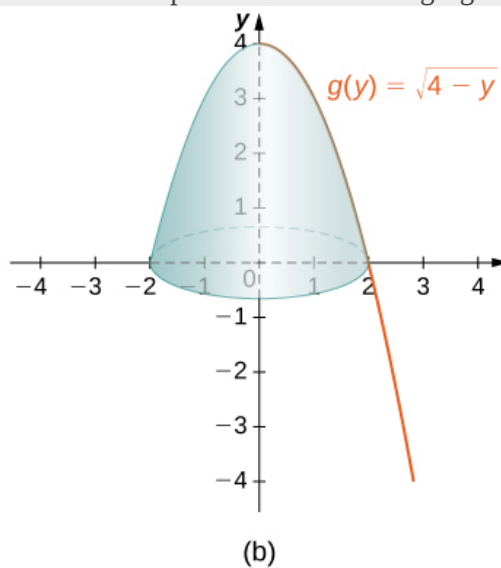
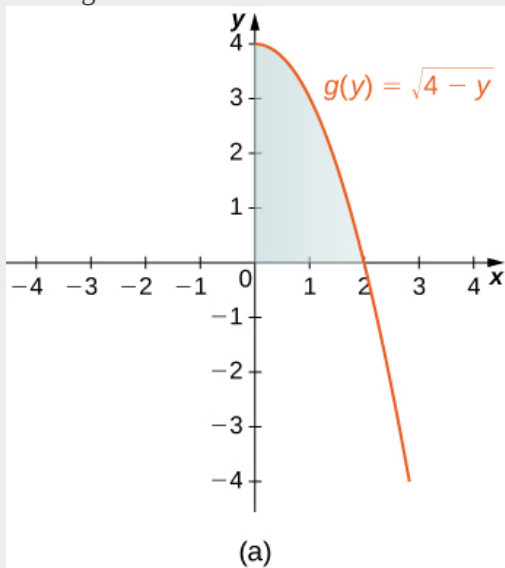
Solution:

[\[link\]](#) shows the function and a representative disk that can be used to estimate the volume. Notice that since we are revolving the function around the y -axis, the disks are horizontal, rather than vertical.



(a) Shown is a thin rectangle between the curve of the function $g(y) = \sqrt{4-y}$ and the y -axis. (b) The rectangle forms a representative disk after revolution around the y -axis.

The region to be revolved and the full solid of revolution are depicted in the following figure.



(a) The region to the left of the function $g(y) = \sqrt{4-y}$ over the y -axis interval $[0, 4]$. (b) The solid of revolution formed by revolving the region about the y -axis.

To find the volume, we integrate with respect to y . We obtain

Equation:

$$\begin{aligned}
 V &= \int_c^d \pi [g(y)]^2 dy \\
 &= \int_0^4 \pi [\sqrt{4-y}]^2 dy = \pi \int_0^4 (4-y) dy \\
 &= \pi \left[4y - \frac{y^2}{2} \right] \Big|_0^4 = 8\pi.
 \end{aligned}$$

The volume is 8π units³.

Note:

Exercise:

Problem:

Use the disk method to find the volume of the solid of revolution generated by rotating the region between the graph of $g(y) = y$ and the y -axis over the interval $[1, 4]$ around the y -axis.

Solution:

21π units³

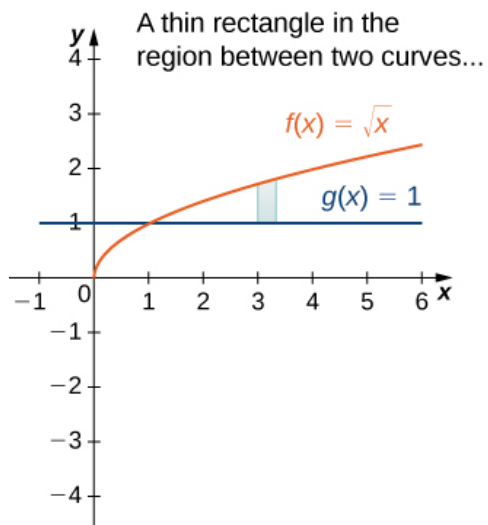
Hint

Use the procedure from [\[link\]](#).

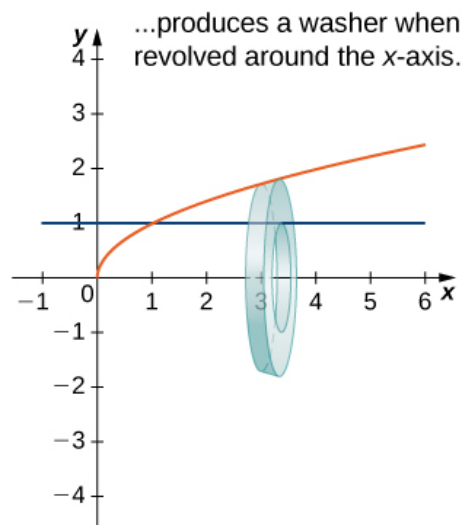
The Washer Method

Some solids of revolution have cavities in the middle; they are not solid all the way to the axis of revolution. Sometimes, this is just a result of the way the region of revolution is shaped with respect to the axis of revolution. In other cases, cavities arise when the region of revolution is defined as the region between the graphs of two functions. A third way this can happen is when an axis of revolution other than the x -axis or y -axis is selected.

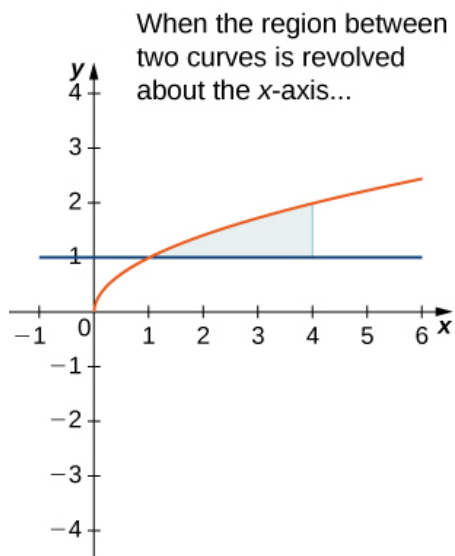
When the solid of revolution has a cavity in the middle, the slices used to approximate the volume are not disks, but washers (disks with holes in the center). For example, consider the region bounded above by the graph of the function $f(x) = \sqrt{x}$ and below by the graph of the function $g(x) = 1$ over the interval $[1, 4]$. When this region is revolved around the x -axis, the result is a solid with a cavity in the middle, and the slices are washers. The graph of the function and a representative washer are shown in [\[link\]](#)(a) and (b). The region of revolution and the resulting solid are shown in [\[link\]](#)(c) and (d).



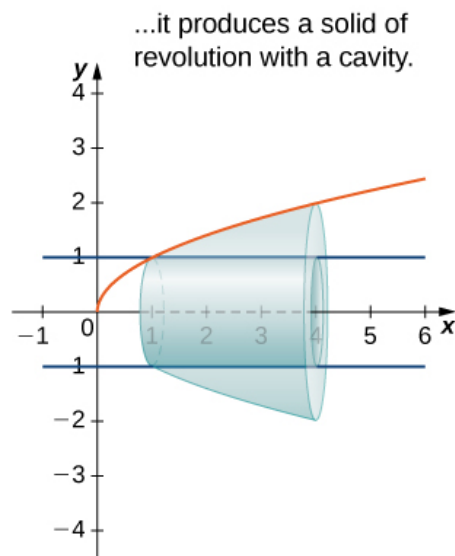
(a)



(b)



(c)



(d)

(a) A thin rectangle in the region between two curves. (b) A representative disk formed by revolving the rectangle about the x -axis. (c) The region between the curves over the given interval. (d) The resulting solid of revolution.

The cross-sectional area, then, is the area of the outer circle less the area of the inner circle. In this case,
Equation:

$$A(x) = \pi(\sqrt{x})^2 - \pi(1)^2 = \pi(x - 1).$$

Then the volume of the solid is

Equation:

$$\begin{aligned} V &= \int_a^b A(x) dx \\ &= \int_1^4 \pi(x-1) dx = \pi \left[\frac{x^2}{2} - x \right] \Big|_1^4 = \frac{9}{2} \pi \text{ units}^3. \end{aligned}$$

Generalizing this process gives the **washer method**.

Note:

Rule: The Washer Method

Suppose $f(x)$ and $g(x)$ are continuous, nonnegative functions such that $f(x) \geq g(x)$ over $[a, b]$. Let R denote the region bounded above by the graph of $f(x)$, below by the graph of $g(x)$, on the left by the line $x = a$, and on the right by the line $x = b$. Then, the volume of the solid of revolution formed by revolving R around the x -axis is given by

Equation:

$$V = \int_a^b \pi \left[(f(x))^2 - (g(x))^2 \right] dx.$$

Example:

Exercise:

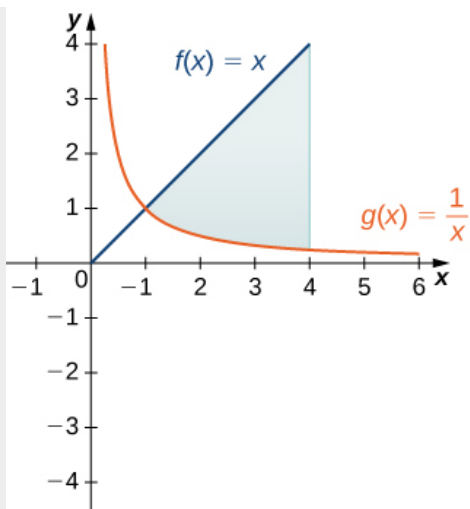
Problem:

Using the Washer Method

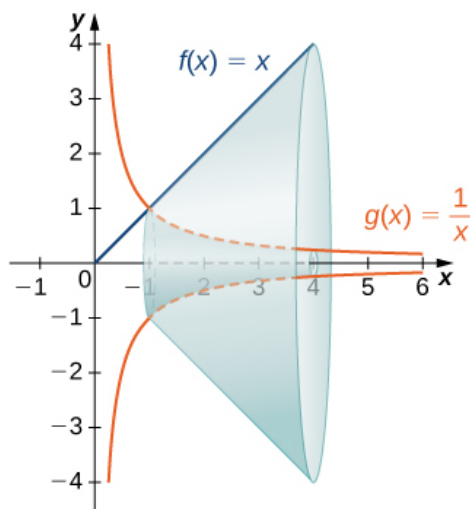
Find the volume of a solid of revolution formed by revolving the region bounded above by the graph of $f(x) = x$ and below by the graph of $g(x) = 1/x$ over the interval $[1, 4]$ around the x -axis.

Solution:

The graphs of the functions and the solid of revolution are shown in the following figure.



(a)



(b)

(a) The region between the graphs of the functions $f(x) = x$ and $g(x) = 1/x$ over the interval $[1, 4]$. (b) Revolving the region about the x -axis generates a solid of revolution with a cavity in the middle.

We have

Equation:

$$\begin{aligned}
 V &= \int_a^b \pi \left[(f(x))^2 - (g(x))^2 \right] dx \\
 &= \pi \int_1^4 \left[x^2 - \left(\frac{1}{x} \right)^2 \right] dx = \pi \left[\frac{x^3}{3} + \frac{1}{x} \right] \Big|_1^4 = \frac{81\pi}{4} \text{ units}^3.
 \end{aligned}$$

Note:

Exercise:

Problem:

Find the volume of a solid of revolution formed by revolving the region bounded by the graphs of $f(x) = \sqrt{x}$ and $g(x) = 1/x$ over the interval $[1, 3]$ around the x -axis.

Solution:

$$\frac{10\pi}{3} \text{ units}^3$$

Hint

Graph the functions to determine which graph forms the upper bound and which graph forms the lower bound, then use the procedure from [\[link\]](#).

As with the disk method, we can also apply the washer method to solids of revolution that result from revolving a region around the y -axis. In this case, the following rule applies.

Note:

Rule: The Washer Method for Solids of Revolution around the y -axis

Suppose $u(y)$ and $v(y)$ are continuous, nonnegative functions such that $v(y) \leq u(y)$ for $y \in [c, d]$. Let Q denote the region bounded on the right by the graph of $u(y)$, on the left by the graph of $v(y)$, below by the line $y = c$, and above by the line $y = d$. Then, the volume of the solid of revolution formed by revolving Q around the y -axis is given by

Equation:

$$V = \int_c^d \pi \left[(u(y))^2 - (v(y))^2 \right] dy.$$

Rather than looking at an example of the washer method with the y -axis as the axis of revolution, we now consider an example in which the axis of revolution is a line other than one of the two coordinate axes. The same general method applies, but you may have to visualize just how to describe the cross-sectional area of the volume.

Example:

Exercise:

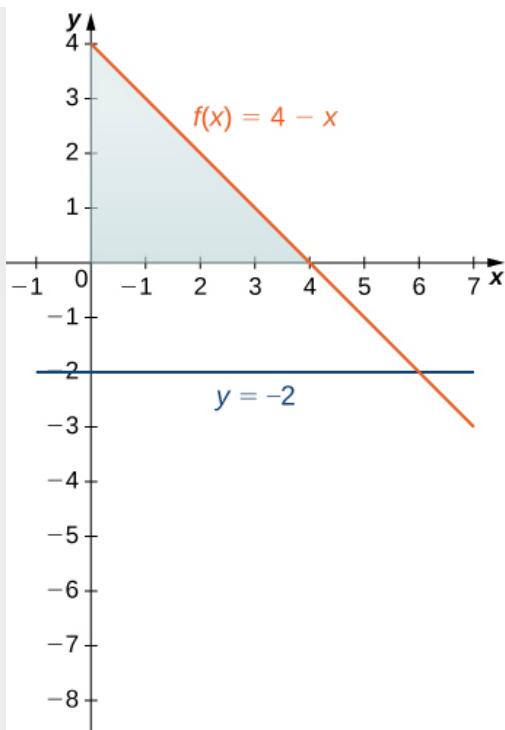
Problem:

The Washer Method with a Different Axis of Revolution

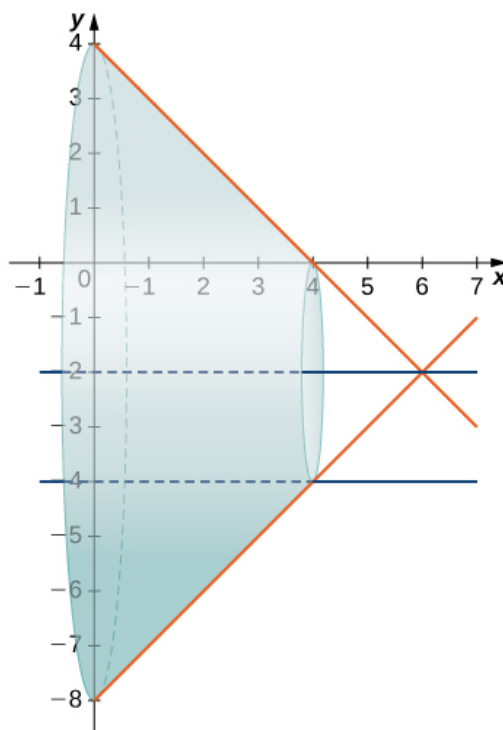
Find the volume of a solid of revolution formed by revolving the region bounded above by $f(x) = 4 - x$ and below by the x -axis over the interval $[0, 4]$ around the line $y = -2$.

Solution:

The graph of the region and the solid of revolution are shown in the following figure.



(a)



(b)

(a) The region between the graph of the function $f(x) = 4 - x$ and the x -axis over the interval $[0, 4]$. (b) Revolving the region about the line $y = -2$ generates a solid of revolution with a cylindrical hole through its middle.

We can't apply the volume formula to this problem directly because the axis of revolution is not one of the coordinate axes. However, we still know that the area of the cross-section is the area of the outer circle less the area of the inner circle. Looking at the graph of the function, we see the radius of the outer circle is given by $f(x) + 2$, which simplifies to

Equation:

$$f(x) + 2 = (4 - x) + 2 = 6 - x.$$

The radius of the inner circle is $g(x) = 2$. Therefore, we have

Equation:

$$\begin{aligned} V &= \int_0^4 \pi \left[(6 - x)^2 - (2)^2 \right] dx \\ &= \pi \int_0^4 (x^2 - 12x + 32) dx = \pi \left[\frac{x^3}{3} - 6x^2 + 32x \right] \Big|_0^4 = \frac{160\pi}{3} \text{ units}^3. \end{aligned}$$

Note:**Exercise:****Problem:**

Find the volume of a solid of revolution formed by revolving the region bounded above by the graph of $f(x) = x + 2$ and below by the x -axis over the interval $[0, 3]$ around the line $y = -1$.

Solution:

$$60\pi \text{ units}^3$$

Hint

Use the procedure from [\[link\]](#).

Key Concepts

- Definite integrals can be used to find the volumes of solids. Using the slicing method, we can find a volume by integrating the cross-sectional area.
- For solids of revolution, the volume slices are often disks and the cross-sections are circles. The method of disks involves applying the method of slicing in the particular case in which the cross-sections are circles, and using the formula for the area of a circle.
- If a solid of revolution has a cavity in the center, the volume slices are washers. With the method of washers, the area of the inner circle is subtracted from the area of the outer circle before integrating.

Key Equations

- **Disk Method along the x -axis**

$$V = \int_a^b \pi [f(x)]^2 dx$$

- **Disk Method along the y -axis**

$$V = \int_c^d \pi [g(y)]^2 dy$$

- **Washer Method**

$$V = \int_a^b \pi [(f(x))^2 - (g(x))^2] dx$$

Exercise:

Problem: Derive the formula for the volume of a sphere using the slicing method.

Exercise:

Problem: Use the slicing method to derive the formula for the volume of a cone.

Exercise:**Problem:**

Use the slicing method to derive the formula for the volume of a tetrahedron with side length a .

Exercise:

Problem: Use the disk method to derive the formula for the volume of a trapezoidal cylinder.

Exercise:

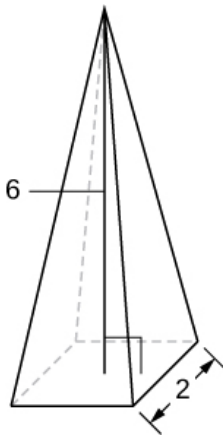
Problem:

Explain when you would use the disk method versus the washer method. When are they interchangeable?

For the following exercises, draw a typical slice and find the volume using the slicing method for the given volume.

Exercise:

Problem: A pyramid with height 6 units and square base of side 2 units, as pictured here.



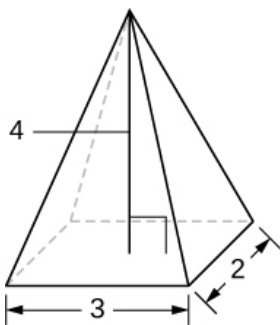
Solution:

8 units³

Exercise:

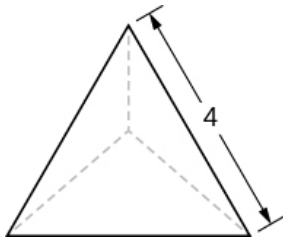
Problem:

A pyramid with height 4 units and a rectangular base with length 2 units and width 3 units, as pictured here.



Exercise:

Problem: A tetrahedron with a base side of 4 units, as seen here.



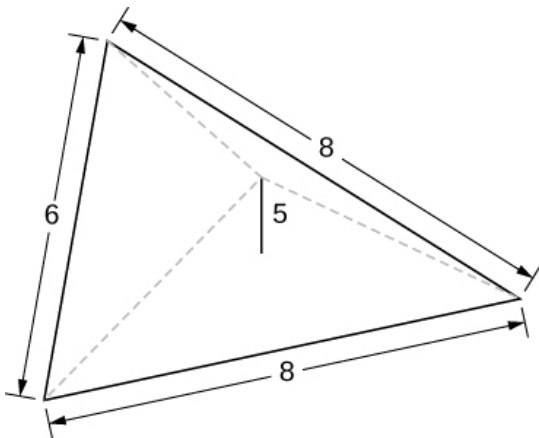
Solution:

$$\frac{32}{3\sqrt{2}} \text{ units}^3$$

Exercise:

Problem:

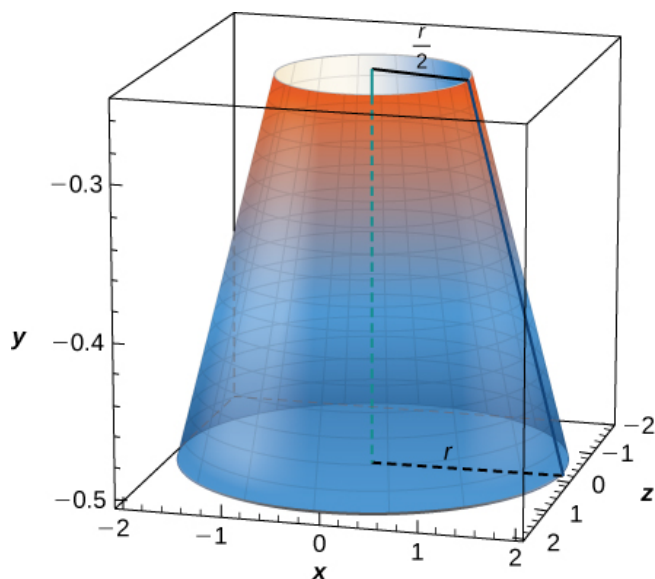
A pyramid with height 5 units, and an isosceles triangular base with lengths of 6 units and 8 units, as seen here.



Exercise:

Problem:

A cone of radius r and height h has a smaller cone of radius $r/2$ and height $h/2$ removed from the top, as seen here. The resulting solid is called a *frustum*.



Solution:

$$\frac{7\pi}{12} h r^2 \text{ units}^3$$

For the following exercises, draw an outline of the solid and find the volume using the slicing method.

Exercise:

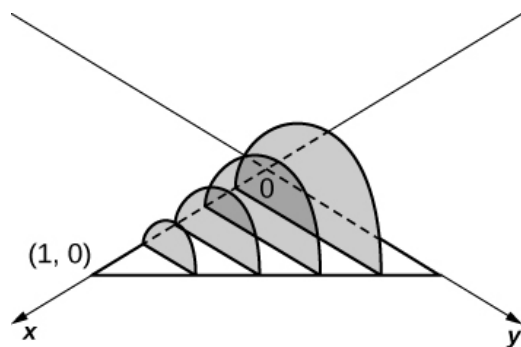
Problem: The base is a circle of radius a . The slices perpendicular to the base are squares.

Exercise:

Problem:

The base is a triangle with vertices $(0, 0)$, $(1, 0)$, and $(0, 1)$. Slices perpendicular to the x -axis are semicircles.

Solution:



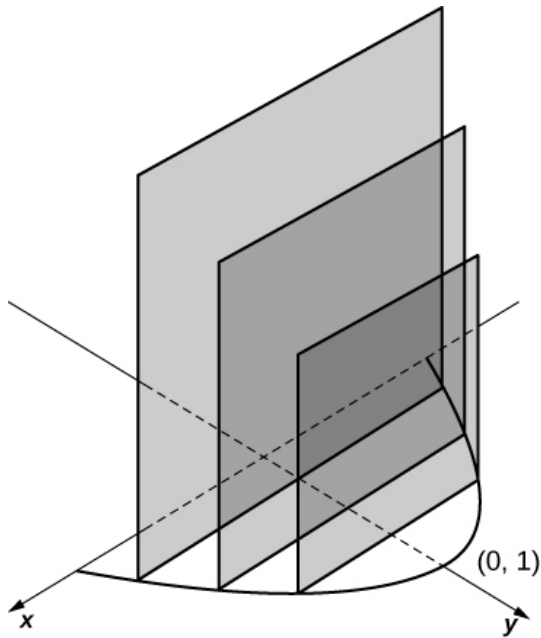
$$\frac{\pi}{24} \text{ units}^3$$

Exercise:**Problem:**

The base is the region under the parabola $y = 1 - x^2$ in the first quadrant. Slices perpendicular to the xy -plane are squares.

Exercise:**Problem:**

The base is the region under the parabola $y = 1 - x^2$ and above the x -axis. Slices perpendicular to the y -axis are squares.

Solution:

2 units³

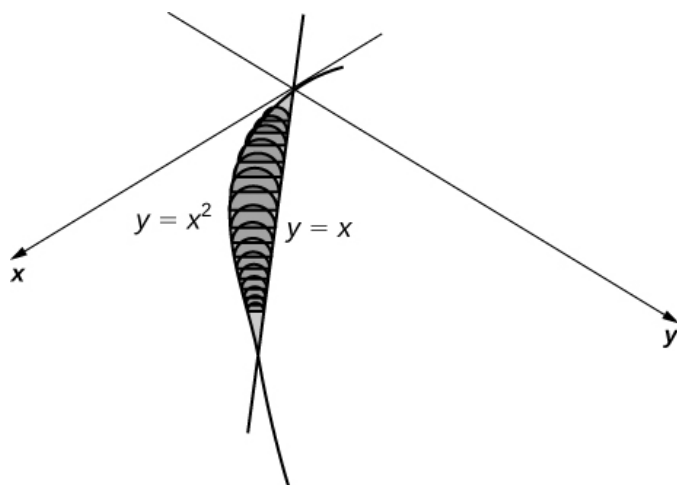
Exercise:**Problem:**

The base is the region enclosed by $y = x^2$ and $y = 9$. Slices perpendicular to the x -axis are right isosceles triangles. The intersection of one of these slices and the base is the leg of the triangle.

Exercise:**Problem:**

The base is the area between $y = x$ and $y = x^2$. Slices perpendicular to the x -axis are semicircles.

Solution:



$$\frac{\pi}{240} \text{ units}^3$$

For the following exercises, draw the region bounded by the curves. Then, use the disk method to find the volume when the region is rotated around the x -axis.

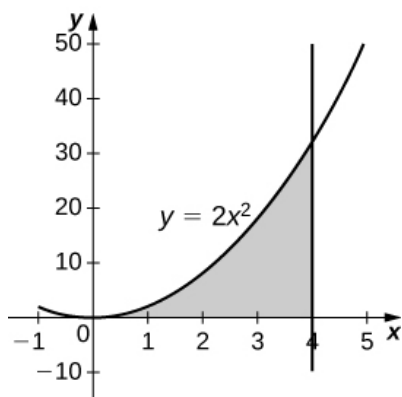
Exercise:

Problem: $x + y = 8$, $x = 0$, and $y = 0$

Exercise:

Problem: $y = 2x^2$, $x = 0$, $x = 4$, and $y = 0$

Solution:



$$\frac{4096\pi}{5} \text{ units}^3$$

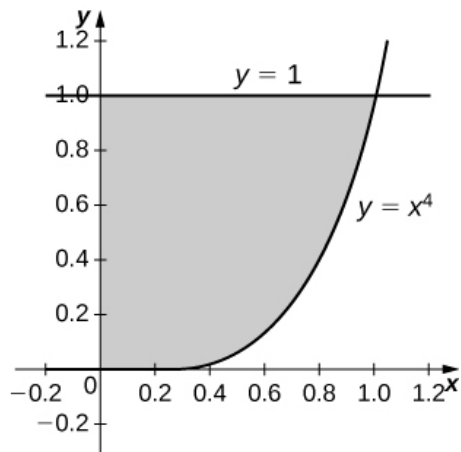
Exercise:

Problem: $y = e^x + 1$, $x = 0$, $x = 1$, and $y = 0$

Exercise:

Problem: $y = x^4$, $x = 0$, and $y = 1$

Solution:



$\frac{8\pi}{9}$ units³

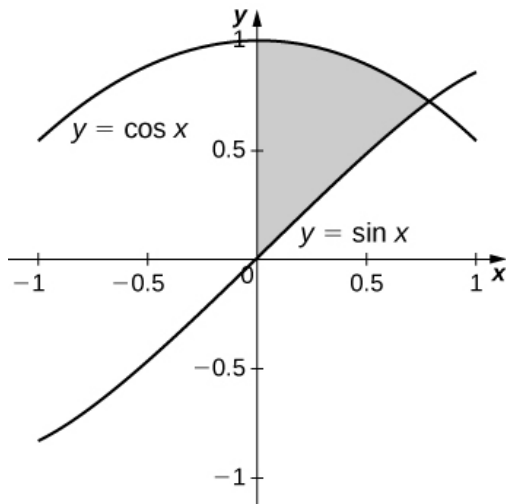
Exercise:

Problem: $y = \sqrt{x}$, $x = 0$, $x = 4$, and $y = 0$

Exercise:

Problem: $y = \sin x$, $y = \cos x$, and $x = 0$

Solution:



$$\frac{\pi}{2} \text{ units}^3$$

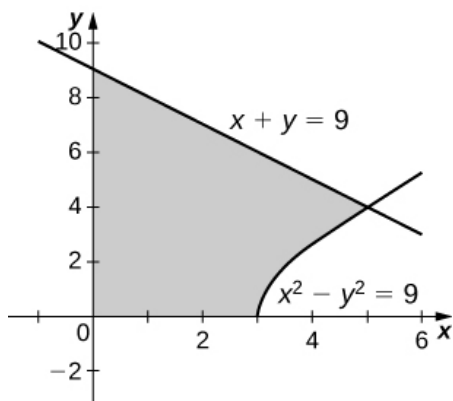
Exercise:

Problem: $y = \frac{1}{x}$, $x = 2$, and $y = 3$

Exercise:

Problem: $x^2 - y^2 = 9$ and $x + y = 9$, $y = 0$ and $x = 0$

Solution:



$$207\pi \text{ units}^3$$

For the following exercises, draw the region bounded by the curves. Then, find the volume when the region is rotated around the y-axis.

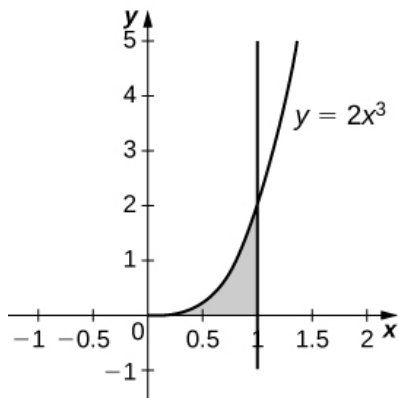
Exercise:

Problem: $y = 4 - \frac{1}{2}x$, $x = 0$, and $y = 0$

Exercise:

Problem: $y = 2x^3$, $x = 0$, $x = 1$, and $y = 0$

Solution:



$$\frac{4\pi}{5} \text{ units}^3$$

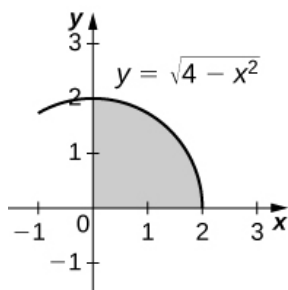
Exercise:

Problem: $y = 3x^2$, $x = 0$, and $y = 3$

Exercise:

Problem: $y = \sqrt{4 - x^2}$, $y = 0$, and $x = 0$

Solution:



$$\frac{16\pi}{3} \text{ units}^3$$

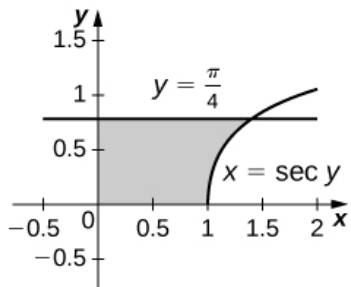
Exercise:

Problem: $y = \frac{1}{\sqrt{x+1}}$, $x = 0$, and $x = 3$

Exercise:

Problem: $x = \sec(y)$ and $y = \frac{\pi}{4}$, $y = 0$ and $x = 0$

Solution:



$\pi \text{ units}^3$

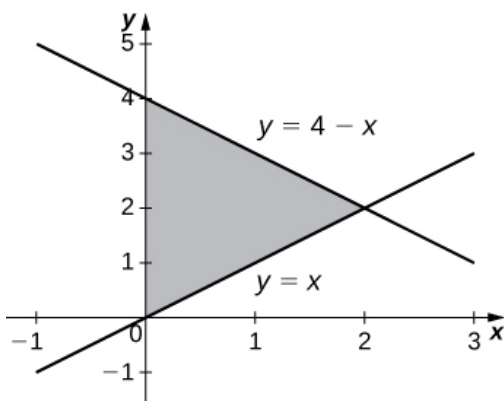
Exercise:

Problem: $y = \frac{1}{x+1}$, $x = 0$, and $x = 2$

Exercise:

Problem: $y = 4 - x$, $y = x$, and $x = 0$

Solution:



$\frac{16\pi}{3} \text{ units}^3$

For the following exercises, draw the region bounded by the curves. Then, find the volume when the region is rotated around the x -axis.

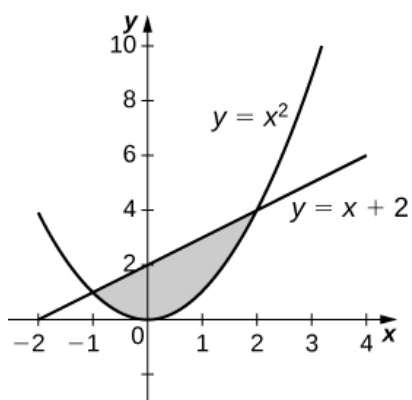
Exercise:

Problem: $y = x + 2$, $y = x + 6$, $x = 0$, and $x = 5$

Exercise:

Problem: $y = x^2$ and $y = x + 2$

Solution:



$$\frac{72\pi}{5} \text{ units}^3$$

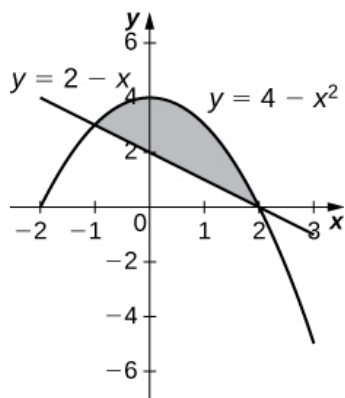
Exercise:

Problem: $x^2 = y^3$ and $x^3 = y^2$

Exercise:

Problem: $y = 4 - x^2$ and $y = 2 - x$

Solution:



$$\frac{108\pi}{5} \text{ units}^3$$

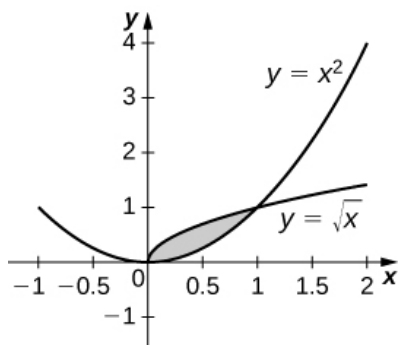
Exercise:

Problem: [T] $y = \cos x$, $y = e^{-x}$, $x = 0$, and $x = 1.2927$

Exercise:

Problem: $y = \sqrt{x}$ and $y = x^2$

Solution:



$$\frac{3\pi}{10} \text{ units}^3$$

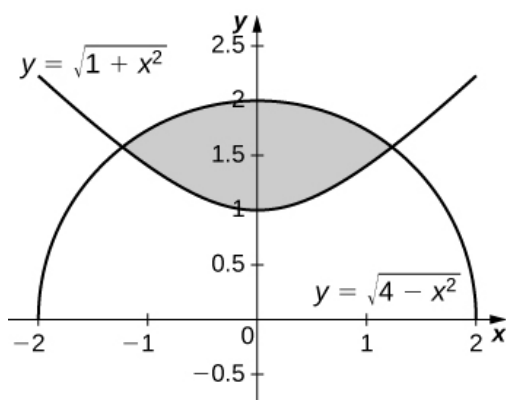
Exercise:

Problem: $y = \sin x$, $y = 5 \sin x$, $x = 0$ and $x = \pi$

Exercise:

Problem: $y = \sqrt{1 + x^2}$ and $y = \sqrt{4 - x^2}$

Solution:



$$2\sqrt{6}\pi \text{ units}^3$$

For the following exercises, draw the region bounded by the curves. Then, use the washer method to find the volume when the region is revolved around the y -axis.

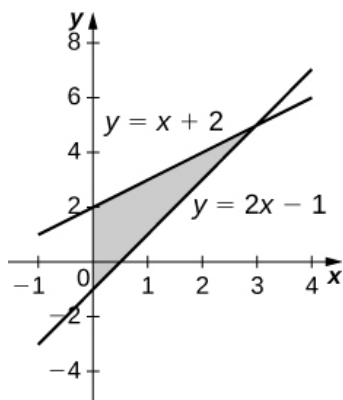
Exercise:

Problem: $y = \sqrt{x}$, $x = 4$, and $y = 0$

Exercise:

Problem: $y = x + 2$, $y = 2x - 1$, and $x = 0$

Solution:



$$9\pi \text{ units}^3$$

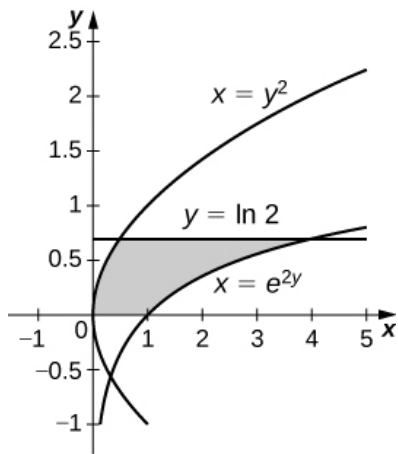
Exercise:

Problem: $y = \sqrt[3]{x}$ and $y = x^3$

Exercise:

Problem: $x = e^{2y}$, $x = y^2$, $y = 0$, and $y = \ln(2)$

Solution:



$$\frac{\pi}{20} (75 - 4 \ln^5(2)) \text{ units}^3$$

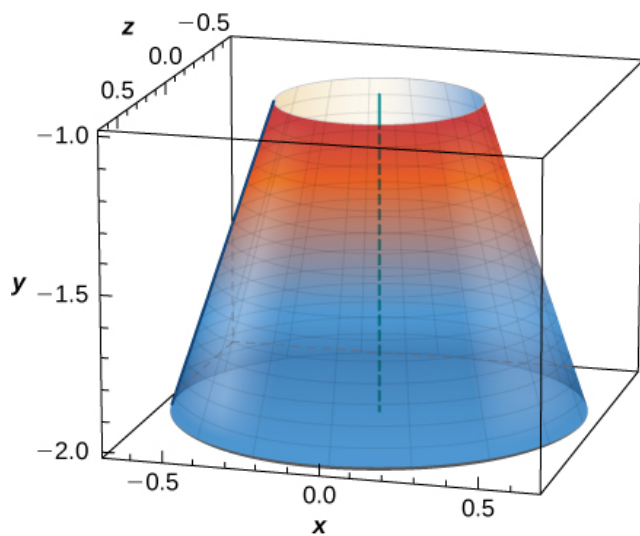
Exercise:

Problem: $x = \sqrt{9 - y^2}$, $x = e^{-y}$, $y = 0$, and $y = 3$

Exercise:

Problem:

Yogurt containers can be shaped like frustums. Rotate the line $y = \frac{1}{m}x$ around the y-axis to find the volume between $y = a$ and $y = b$.



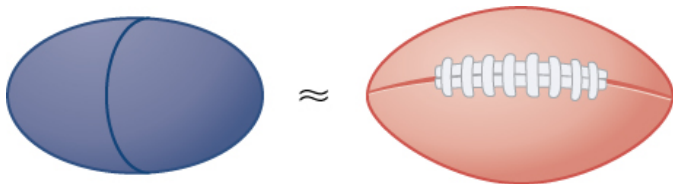
Solution:

$$\frac{m^2\pi}{3}(b^3 - a^3) \text{ units}^3$$

Exercise:

Problem:

Rotate the ellipse $(x^2/a^2) + (y^2/b^2) = 1$ around the x-axis to approximate the volume of a football, as seen here.



Exercise:

Problem:

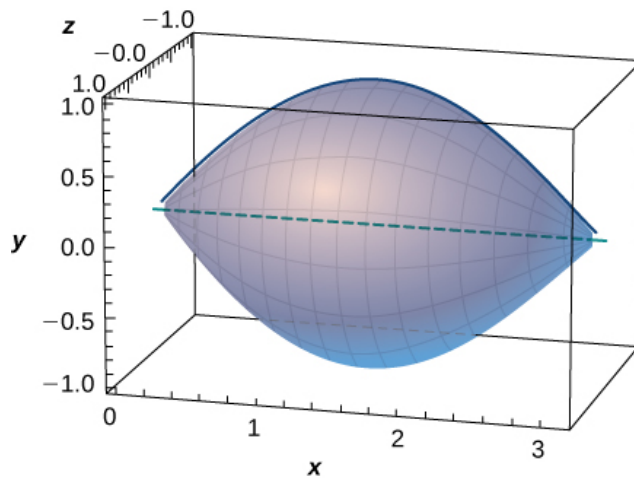
Rotate the ellipse $(x^2/a^2) + (y^2/b^2) = 1$ around the y -axis to approximate the volume of a football.

Solution:

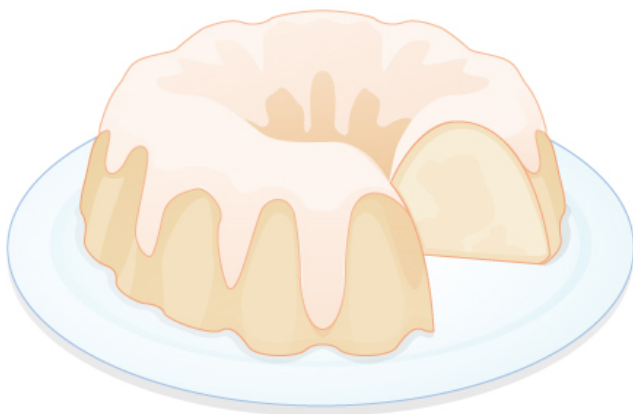
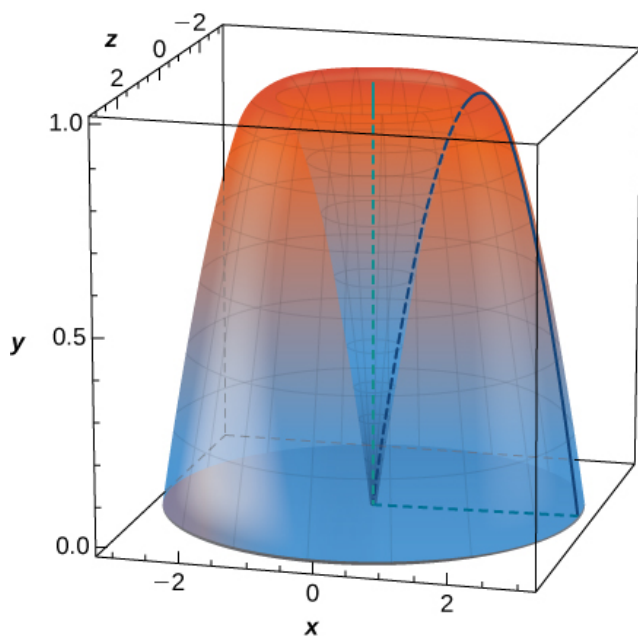
$$\frac{4a^2b\pi}{3} \text{ units}^3$$

Exercise:**Problem:**

A better approximation of the volume of a football is given by the solid that comes from rotating $y = \sin x$ around the x -axis from $x = 0$ to $x = \pi$. What is the volume of this football approximation, as seen here?

**Exercise:****Problem:**

What is the volume of the Bundt cake that comes from rotating $y = \sin x$ around the y -axis from $x = 0$ to $x = \pi$?



Solution:

$$2\pi^2 \text{ units}^3$$

For the following exercises, find the volume of the solid described.

Exercise:

Problem:

The base is the region between $y = x$ and $y = x^2$. Slices perpendicular to the x -axis are semicircles.

Exercise:

Problem:

The base is the region enclosed by the generic ellipse $(x^2/a^2) + (y^2/b^2) = 1$. Slices perpendicular to the x -axis are semicircles.

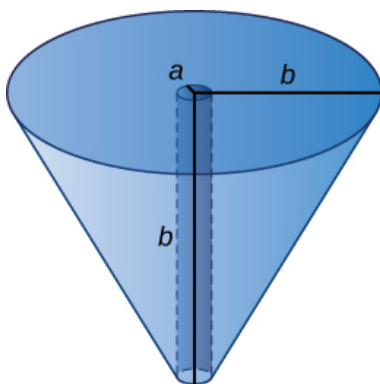
Solution:

$$\frac{2ab^2\pi}{3} \text{ units}^3$$

Exercise:

Problem:

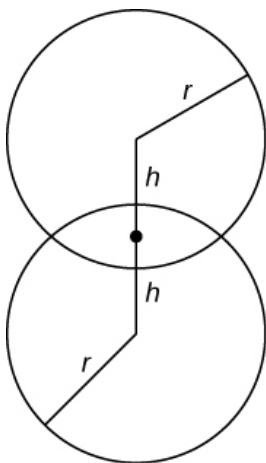
Bore a hole of radius a down the axis of a right cone and through the base of radius b , as seen here.



Exercise:

Problem:

Find the volume common to two spheres of radius r with centers that are $2h$ apart, as shown here.

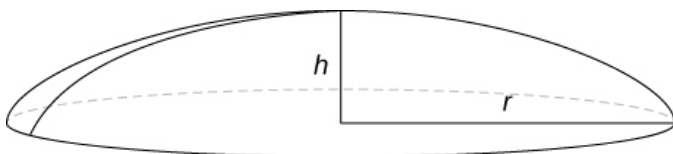


Solution:

$$\frac{\pi}{12}(r+h)^2(6r-h) \text{ units}^3$$

Exercise:

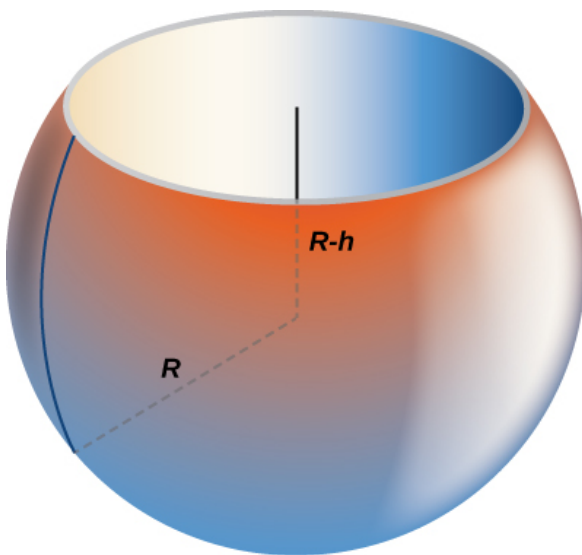
Problem: Find the volume of a spherical cap of height h and radius r where $h < r$, as seen here.



Exercise:

Problem:

Find the volume of a sphere of radius R with a cap of height h removed from the top, as seen here.



Solution:

$$\frac{\pi}{3}(h + R)(h - 2R)^2 \text{ units}^3$$

Glossary

cross-section

the intersection of a plane and a solid object

disk method

a special case of the slicing method used with solids of revolution when the slices are disks

slicing method

a method of calculating the volume of a solid that involves cutting the solid into pieces, estimating the volume of each piece, then adding these estimates to arrive at an estimate of the total volume; as the number of slices goes to infinity, this estimate becomes an integral that gives the exact value of the volume

solid of revolution

a solid generated by revolving a region in a plane around a line in that plane

washer method

a special case of the slicing method used with solids of revolution when the slices are washers

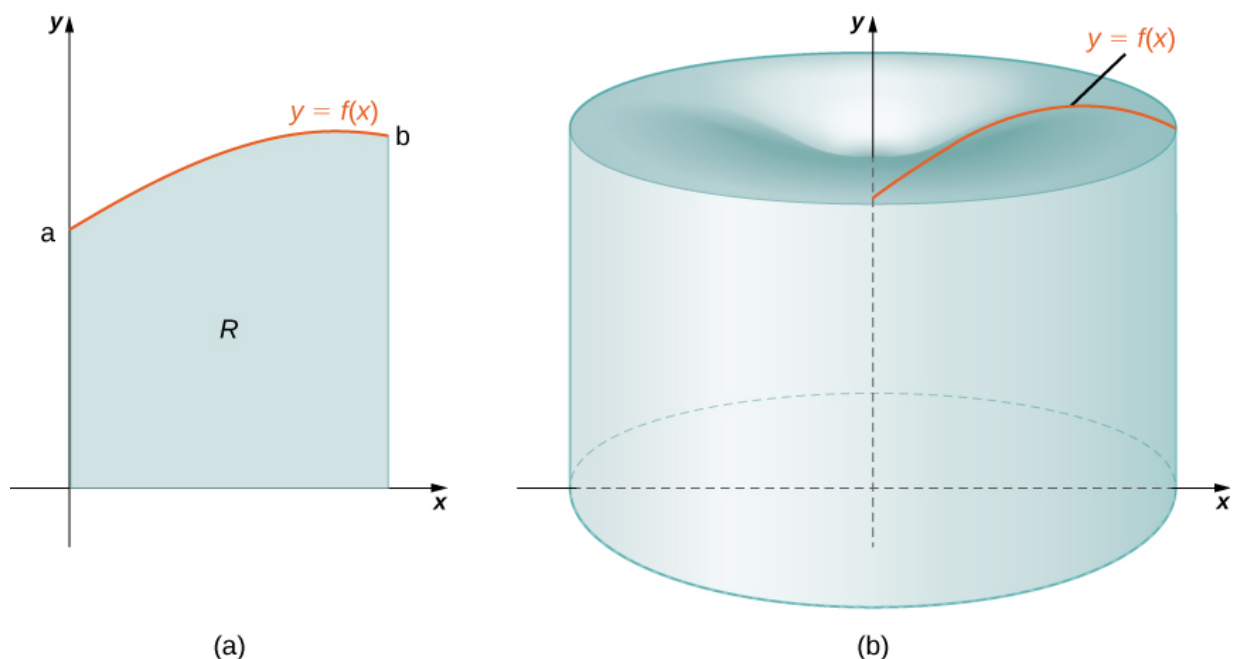
Volumes of Revolution: Cylindrical Shells

- Calculate the volume of a solid of revolution by using the method of cylindrical shells.
- Compare the different methods for calculating a volume of revolution.

In this section, we examine the method of cylindrical shells, the final method for finding the volume of a solid of revolution. We can use this method on the same kinds of solids as the disk method or the washer method; however, with the disk and washer methods, we integrate along the coordinate axis parallel to the axis of revolution. With the method of cylindrical shells, we integrate along the coordinate axis *perpendicular* to the axis of revolution. The ability to choose which variable of integration we want to use can be a significant advantage with more complicated functions. Also, the specific geometry of the solid sometimes makes the method of using cylindrical shells more appealing than using the washer method. In the last part of this section, we review all the methods for finding volume that we have studied and lay out some guidelines to help you determine which method to use in a given situation.

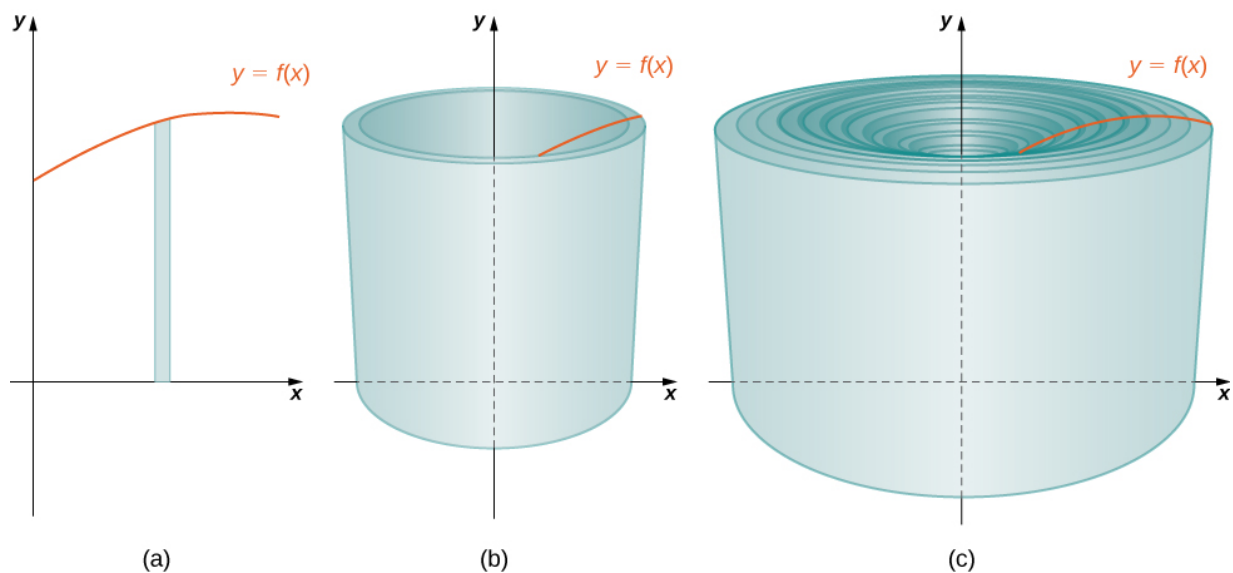
The Method of Cylindrical Shells

Again, we are working with a solid of revolution. As before, we define a region R , bounded above by the graph of a function $y = f(x)$, below by the x -axis, and on the left and right by the lines $x = a$ and $x = b$, respectively, as shown in [\[link\]](#)(a). We then revolve this region around the y -axis, as shown in [\[link\]](#)(b). Note that this is different from what we have done before. Previously, regions defined in terms of functions of x were revolved around the x -axis or a line parallel to it.



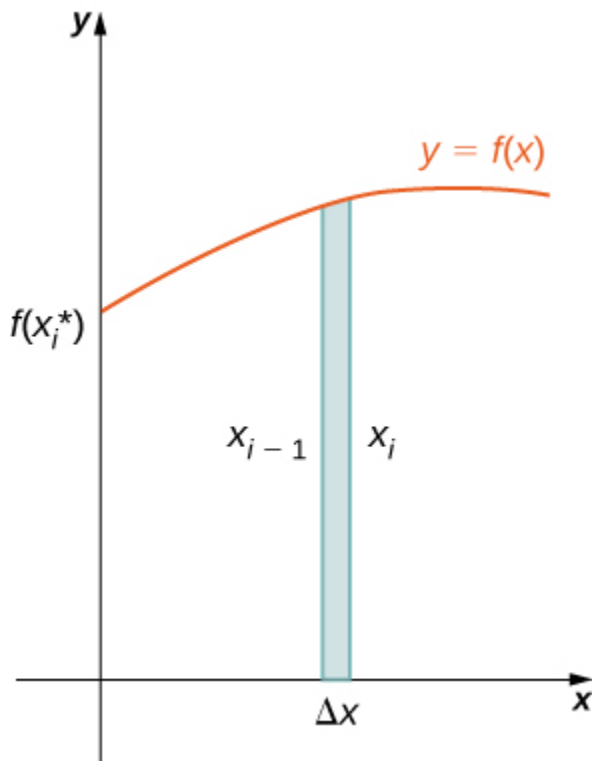
(a) A region bounded by the graph of a function of x . (b) The solid of revolution formed when the region is revolved around the y -axis.

As we have done many times before, partition the interval $[a, b]$ using a regular partition, $P = \{x_0, x_1, \dots, x_n\}$ and, for $i = 1, 2, \dots, n$, choose a point $x_i^* \in [x_{i-1}, x_i]$. Then, construct a rectangle over the interval $[x_{i-1}, x_i]$ of height $f(x_i^*)$ and width Δx . A representative rectangle is shown in [\[link\]](#)(a). When that rectangle is revolved around the y -axis, instead of a disk or a washer, we get a cylindrical shell, as shown in the following figure.



(a) A representative rectangle. (b) When this rectangle is revolved around the y -axis, the result is a cylindrical shell. (c) When we put all the shells together, we get an approximation of the original solid.

To calculate the volume of this shell, consider [\[link\]](#).



Calculating the volume of the shell.

The shell is a cylinder, so its volume is the cross-sectional area multiplied by the height of the cylinder. The cross-sections are annuli (ring-shaped regions—essentially, circles with a hole in the center), with outer radius x_i and inner radius x_{i-1} . Thus, the cross-sectional area is $\pi x_i^2 - \pi x_{i-1}^2$. The height of the cylinder is $f(x_i^*)$. Then the volume of the shell is

Equation:

$$\begin{aligned}
 V_{\text{shell}} &= f(x_i^*)(\pi x_i^2 - \pi x_{i-1}^2) \\
 &= \pi f(x_i^*)(x_i^2 - x_{i-1}^2) \\
 &= \pi f(x_i^*)(x_i + x_{i-1})(x_i - x_{i-1}) \\
 &= 2\pi f(x_i^*)\left(\frac{x_i + x_{i-1}}{2}\right)(x_i - x_{i-1}).
 \end{aligned}$$

Note that $x_i - x_{i-1} = \Delta x$, so we have

Equation:

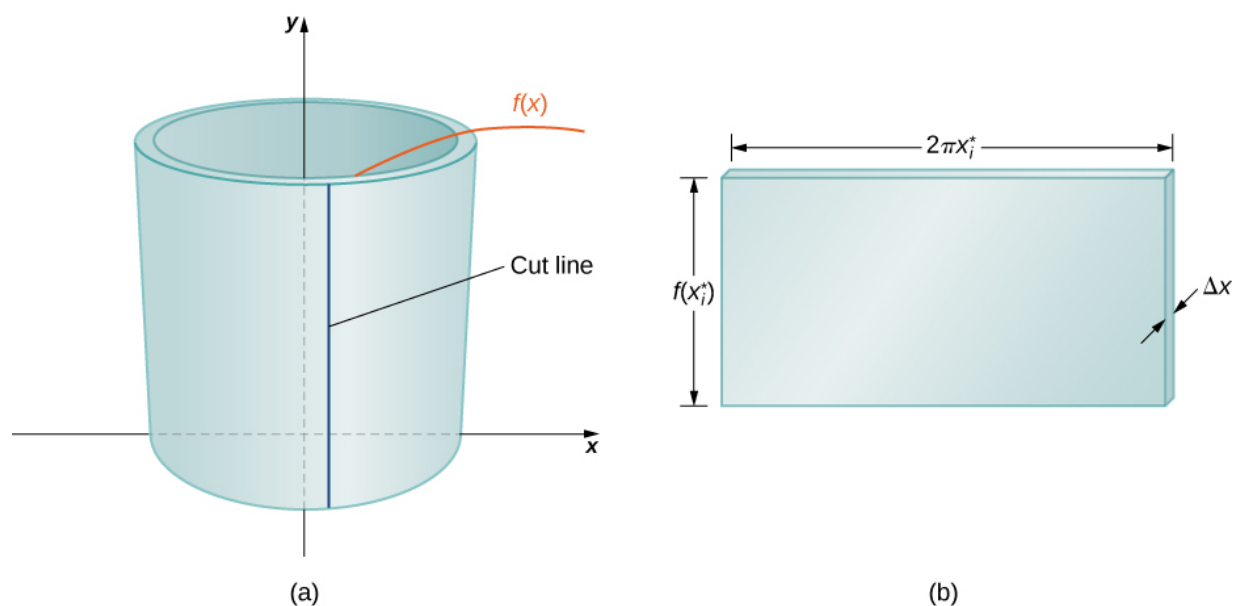
$$V_{\text{shell}} = 2\pi f(x_i^*) \left(\frac{x_i + x_{i-1}}{2} \right) \Delta x.$$

Furthermore, $\frac{x_i + x_{i-1}}{2}$ is both the midpoint of the interval $[x_{i-1}, x_i]$ and the average radius of the shell, and we can approximate this by x_i^* . We then have

Equation:

$$V_{\text{shell}} \approx 2\pi f(x_i^*) x_i^* \Delta x.$$

Another way to think of this is to think of making a vertical cut in the shell and then opening it up to form a flat plate ([link](#)).



(a) Make a vertical cut in a representative shell. (b) Open the shell up to form a flat plate.

In reality, the outer radius of the shell is greater than the inner radius, and hence the back edge of the plate would be slightly longer than the front edge of the plate. However, we can approximate the flattened shell by a flat plate of height $f(x_i^*)$, width $2\pi x_i^*$, and thickness Δx ([link](#)). The volume of the shell, then, is approximately the volume of the flat plate. Multiplying the height, width, and depth of the plate, we get

Equation:

$$V_{\text{shell}} \approx f(x_i^*) (2\pi x_i^*) \Delta x,$$

which is the same formula we had before.

To calculate the volume of the entire solid, we then add the volumes of all the shells and obtain

Equation:

$$V \approx \sum_{i=1}^n (2\pi x_i^* f(x_i^*) \Delta x).$$

Here we have another Riemann sum, this time for the function $2\pi x f(x)$. Taking the limit as $n \rightarrow \infty$ gives us

Equation:

$$V = \lim_{n \rightarrow \infty} \sum_{i=1}^n (2\pi x_i^* f(x_i^*) \Delta x) = \int_a^b (2\pi x f(x)) dx.$$

This leads to the following rule for the **method of cylindrical shells**.

Note:

Rule: The Method of Cylindrical Shells

Let $f(x)$ be continuous and nonnegative. Define R as the region bounded above by the graph of $f(x)$, below by the x -axis, on the left by the line $x = a$, and on the right by the line $x = b$. Then the volume of the solid of revolution formed by revolving R around the y -axis is given by

Equation:

$$V = \int_a^b (2\pi x f(x)) dx.$$

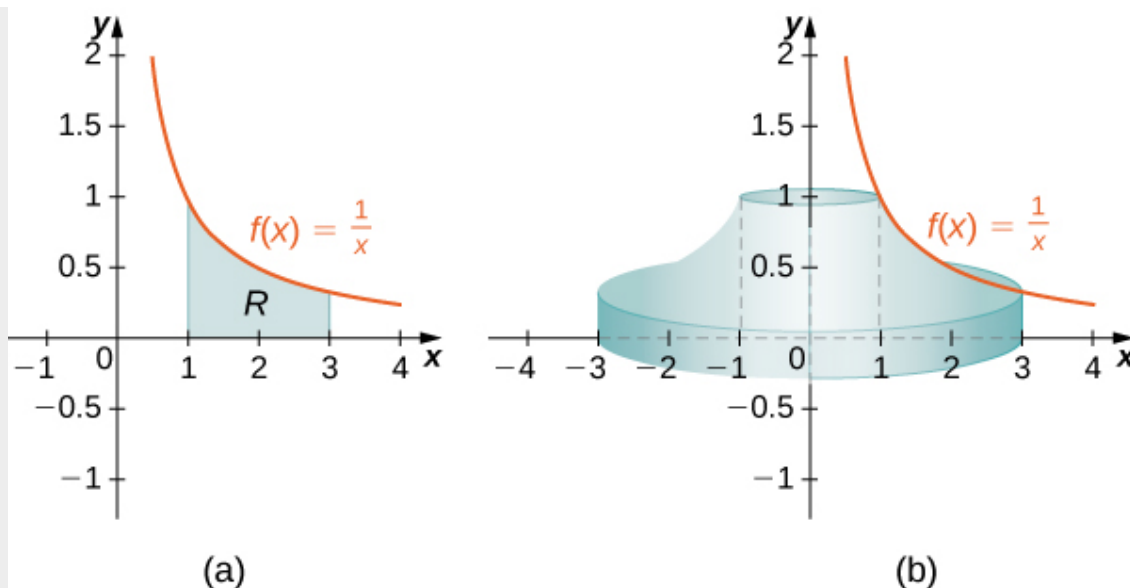
Now let's consider an example.

Example:**Exercise:****Problem:****The Method of Cylindrical Shells 1**

Define R as the region bounded above by the graph of $f(x) = 1/x$ and below by the x -axis over the interval $[1, 3]$. Find the volume of the solid of revolution formed by revolving R around the y -axis.

Solution:

First we must graph the region R and the associated solid of revolution, as shown in the following figure.



(a) The region R under the graph of $f(x) = 1/x$ over the interval $[1, 3]$. (b) The solid of revolution generated by revolving R about the y -axis.

Then the volume of the solid is given by

Equation:

$$\begin{aligned}
 V &= \int_a^b (2\pi x f(x)) dx \\
 &= \int_1^3 \left(2\pi x \left(\frac{1}{x} \right) \right) dx \\
 &= \int_1^3 2\pi dx = 2\pi x \Big|_1^3 = 4\pi \text{ units}^3.
 \end{aligned}$$

Note:

Exercise:

Problem:

Define R as the region bounded above by the graph of $f(x) = x^2$ and below by the x -axis over the interval $[1, 2]$. Find the volume of the solid of revolution formed by revolving R around the y -axis.

Solution:

$$\frac{15\pi}{2} \text{ units}^3$$

Hint

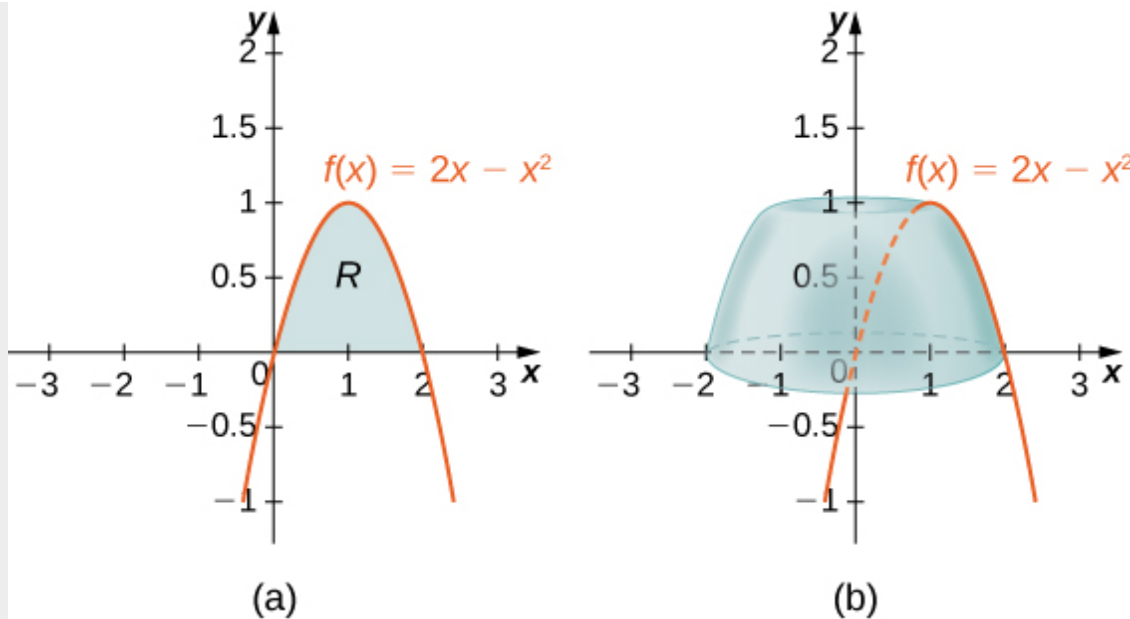
Use the procedure from [\[link\]](#).

Example:**Exercise:****Problem:****The Method of Cylindrical Shells 2**

Define R as the region bounded above by the graph of $f(x) = 2x - x^2$ and below by the x -axis over the interval $[0, 2]$. Find the volume of the solid of revolution formed by revolving R around the y -axis.

Solution:

First graph the region R and the associated solid of revolution, as shown in the following figure.



(a) The region R under the graph of $f(x) = 2x - x^2$ over the interval $[0, 2]$. (b) The volume of revolution obtained by revolving R about the y -axis.

Then the volume of the solid is given by

Equation:

$$\begin{aligned}
 V &= \int_a^b (2\pi x f(x)) dx \\
 &= \int_0^2 (2\pi x (2x - x^2)) dx = 2\pi \int_0^2 (2x^2 - x^3) dx \\
 &= 2\pi \left[\frac{2x^3}{3} - \frac{x^4}{4} \right] \bigg|_0^2 = \frac{8\pi}{3} \text{ units}^3.
 \end{aligned}$$

Note:

Exercise:

Problem:

Define R as the region bounded above by the graph of $f(x) = 3x - x^2$ and below by the x -axis over the interval $[0, 2]$. Find the volume of the solid of revolution formed by revolving R around the y -axis.

Solution:

$8\pi \text{ units}^3$

Hint

Use the process from [\[link\]](#).

As with the disk method and the washer method, we can use the method of cylindrical shells with solids of revolution, revolved around the x -axis, when we want to integrate with respect to y . The analogous rule for this type of solid is given here.

Note:

Rule: The Method of Cylindrical Shells for Solids of Revolution around the x -axis

Let $g(y)$ be continuous and nonnegative. Define Q as the region bounded on the right by the graph of $g(y)$, on the left by the y -axis, below by the line $y = c$, and above by the line $y = d$. Then, the volume of the solid of revolution formed by revolving Q around the x -axis is given by

Equation:

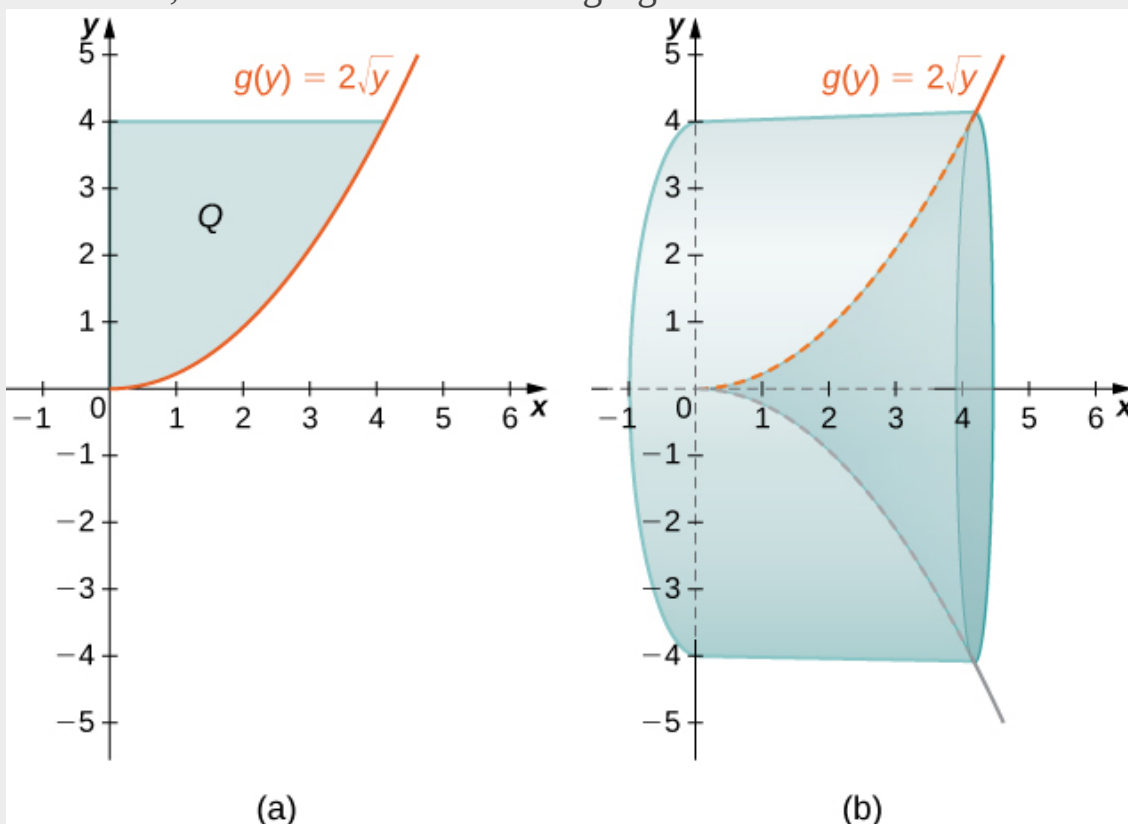
$$V = \int_c^d (2\pi y g(y)) dy.$$

Example:**Exercise:****Problem:****The Method of Cylindrical Shells for a Solid Revolved around the x -axis**

Define Q as the region bounded on the right by the graph of $g(y) = 2\sqrt{y}$ and on the left by the y -axis for $y \in [0, 4]$. Find the volume of the solid of revolution formed by revolving Q around the x -axis.

Solution:

First, we need to graph the region Q and the associated solid of revolution, as shown in the following figure.



- (a) The region Q to the left of the function $g(y)$ over the interval $[0, 4]$. (b) The solid of revolution generated by revolving Q around the x -axis.

Label the shaded region Q . Then the volume of the solid is given by
Equation:

$$\begin{aligned} V &= \int_c^d (2\pi y g(y)) dy \\ &= \int_0^4 (2\pi y (2\sqrt{y})) dy = 4\pi \int_0^4 y^{3/2} dy \\ &= 4\pi \left[\frac{2y^{5/2}}{5} \right] \Big|_0^4 = \frac{256\pi}{5} \text{ units}^3. \end{aligned}$$

Note:

Exercise:

Problem:

Define Q as the region bounded on the right by the graph of $g(y) = 3/y$ and on the left by the y -axis for $y \in [1, 3]$. Find the volume of the solid of revolution formed by revolving Q around the x -axis.

Solution:

$$12\pi \text{ units}^3$$

Hint

Use the process from [\[link\]](#).

For the next example, we look at a solid of revolution for which the graph of a function is revolved around a line other than one of the two coordinate

axes. To set this up, we need to revisit the development of the method of cylindrical shells. Recall that we found the volume of one of the shells to be given by

Equation:

$$\begin{aligned} V_{\text{shell}} &= f(x_i^*)(\pi x_i^2 - \pi x_{i-1}^2) \\ &= \pi f(x_i^*)(x_i^2 - x_{i-1}^2) \\ &= \pi f(x_i^*)(x_i + x_{i-1})(x_i - x_{i-1}) \\ &= 2\pi f(x_i^*)\left(\frac{x_i + x_{i-1}}{2}\right)(x_i - x_{i-1}). \end{aligned}$$

This was based on a shell with an outer radius of x_i and an inner radius of x_{i-1} . If, however, we rotate the region around a line other than the y -axis, we have a different outer and inner radius. Suppose, for example, that we rotate the region around the line $x = -k$, where k is some positive constant. Then, the outer radius of the shell is $x_i + k$ and the inner radius of the shell is $x_{i-1} + k$. Substituting these terms into the expression for volume, we see that when a plane region is rotated around the line $x = -k$, the volume of a shell is given by

Equation:

$$\begin{aligned} V_{\text{shell}} &= 2\pi f(x_i^*)\left(\frac{(x_i + k) + (x_{i-1} + k)}{2}\right)((x_i + k) - (x_{i-1} + k)) \\ &= 2\pi f(x_i^*)\left(\left(\frac{x_i + x_{i-1}}{2}\right) + k\right)\Delta x. \end{aligned}$$

As before, we notice that $\frac{x_i + x_{i-1}}{2}$ is the midpoint of the interval $[x_{i-1}, x_i]$ and can be approximated by x_i^* . Then, the approximate volume of the shell is

Equation:

$$V_{\text{shell}} \approx 2\pi \left(x_i^* + k\right) f(x_i^*) \Delta x.$$

The remainder of the development proceeds as before, and we see that

Equation:

$$V = \int_a^b (2\pi (x + k) f(x)) dx.$$

We could also rotate the region around other horizontal or vertical lines, such as a vertical line in the right half plane. In each case, the volume formula must be adjusted accordingly. Specifically, the x -term in the integral must be replaced with an expression representing the radius of a shell. To see how this works, consider the following example.

Example:

Exercise:

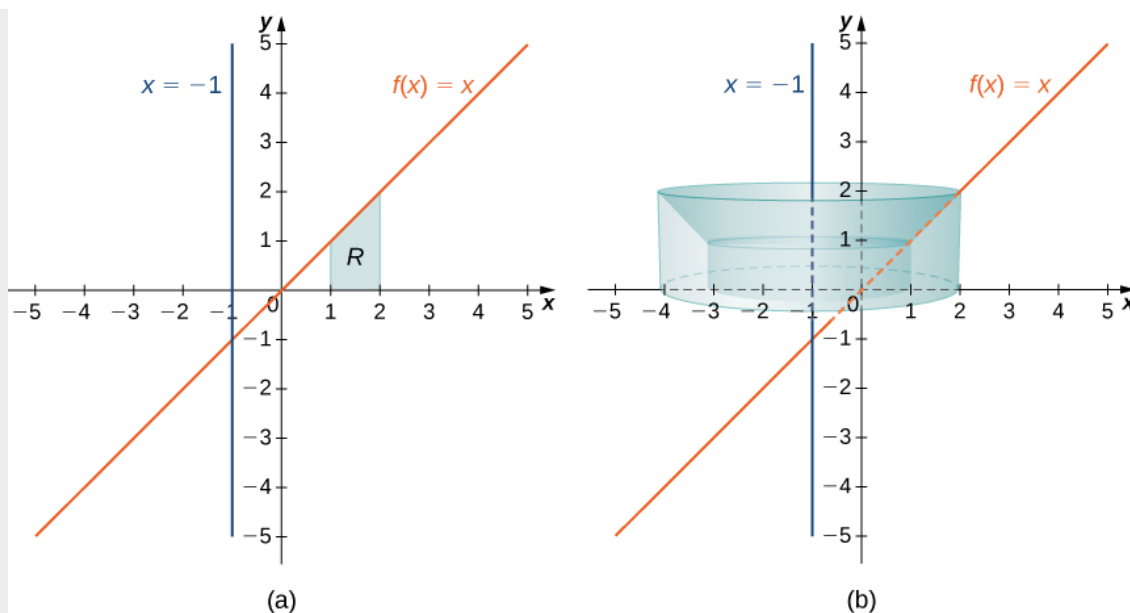
Problem:

A Region of Revolution Revolved around a Line

Define R as the region bounded above by the graph of $f(x) = x$ and below by the x -axis over the interval $[1, 2]$. Find the volume of the solid of revolution formed by revolving R around the line $x = -1$.

Solution:

First, graph the region R and the associated solid of revolution, as shown in the following figure.



(a) The region R between the graph of $f(x)$ and the x -axis over the interval $[1, 2]$. (b) The solid of revolution generated by revolving R around the line $x = -1$.

Note that the radius of a shell is given by $x + 1$. Then the volume of the solid is given by

Equation:

$$\begin{aligned}
 V &= \int_1^2 (2\pi (x + 1) f(x)) dx \\
 &= \int_1^2 (2\pi (x + 1) x) dx = 2\pi \int_1^2 (x^2 + x) dx \\
 &= 2\pi \left[\frac{x^3}{3} + \frac{x^2}{2} \right] \Big|_1^2 = \frac{23\pi}{3} \text{ units}^3.
 \end{aligned}$$

Note:

Exercise:

Problem:

Define R as the region bounded above by the graph of $f(x) = x^2$ and below by the x -axis over the interval $[0, 1]$. Find the volume of the solid of revolution formed by revolving R around the line $x = -2$.

Solution:

$$\frac{11\pi}{6} \text{ units}^3$$

Hint

Use the process from [\[link\]](#).

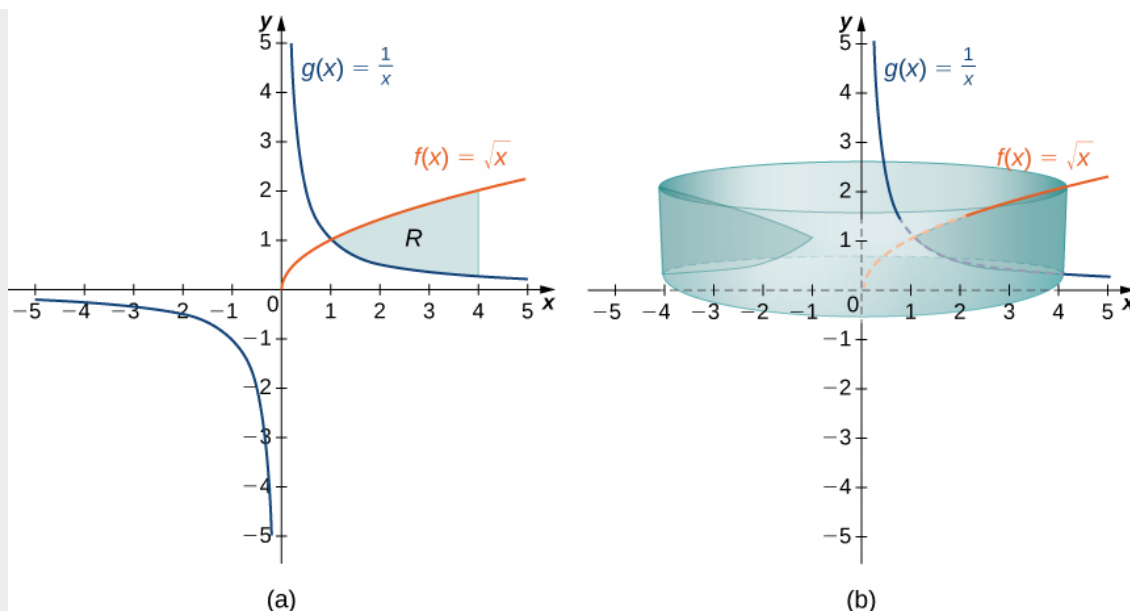
For our final example in this section, let's look at the volume of a solid of revolution for which the region of revolution is bounded by the graphs of two functions.

Example:**Exercise:****Problem:****A Region of Revolution Bounded by the Graphs of Two Functions**

Define R as the region bounded above by the graph of the function $f(x) = \sqrt{x}$ and below by the graph of the function $g(x) = 1/x$ over the interval $[1, 4]$. Find the volume of the solid of revolution generated by revolving R around the y -axis.

Solution:

First, graph the region R and the associated solid of revolution, as shown in the following figure.



- (a) The region R between the graph of $f(x)$ and the graph of $g(x)$ over the interval $[1, 4]$. (b) The solid of revolution generated by revolving R around the y -axis.

Note that the axis of revolution is the y -axis, so the radius of a shell is given simply by x . We don't need to make any adjustments to the x -term of our integrand. The height of a shell, though, is given by $f(x) - g(x)$, so in this case we need to adjust the $f(x)$ term of the integrand. Then the volume of the solid is given by

Equation:

$$\begin{aligned}
 V &= \int_1^4 (2\pi x (f(x) - g(x))) dx \\
 &= \int_1^4 \left(2\pi x \left(\sqrt{x} - \frac{1}{x} \right) \right) dx = 2\pi \int_1^4 \left(x^{3/2} - 1 \right) dx \\
 &= 2\pi \left[\frac{2x^{5/2}}{5} - x \right] \Big|_1^4 = \frac{94\pi}{5} \text{ units}^3.
 \end{aligned}$$

Note:

Exercise:

Problem:

Define R as the region bounded above by the graph of $f(x) = x$ and below by the graph of $g(x) = x^2$ over the interval $[0, 1]$. Find the volume of the solid of revolution formed by revolving R around the y -axis.

Solution:

$$\frac{\pi}{6} \text{ units}^3$$

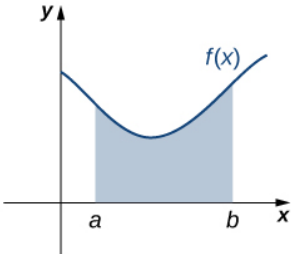
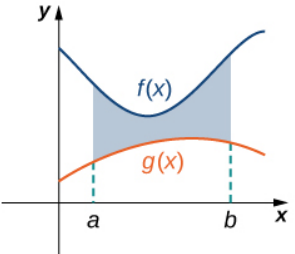
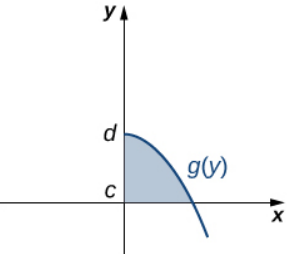
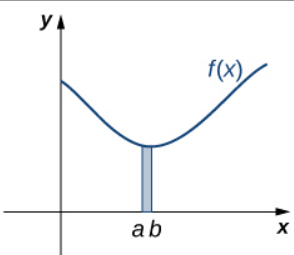
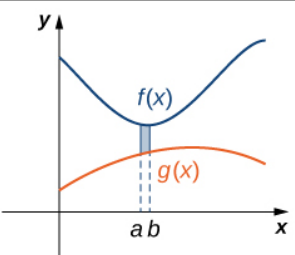
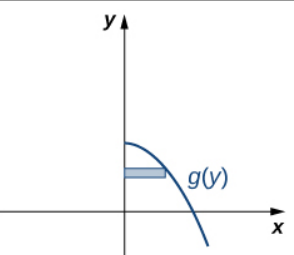
Hint

Use the process from [\[link\]](#).

Which Method Should We Use?

We have studied several methods for finding the volume of a solid of revolution, but how do we know which method to use? It often comes down to a choice of which integral is easiest to evaluate. [\[link\]](#) describes the different approaches for solids of revolution around the x -axis. It's up to you to develop the analogous table for solids of revolution around the y -axis.

Comparing the Methods for Finding the Volume of a Solid Revolution around the x-axis

Compare	Disk Method	Washer Method	Shell Method
Volume formula	$V = \int_a^b \pi [f(x)]^2 dx$	$V = \int_a^b \pi [(f(x))^2 - (g(x))^2] dx$	$V = \int_c^d 2\pi y g(y) dy$
Solid	No cavity in the center	Cavity in the center	With or without a cavity in the center
Interval to partition	$[a, b]$ on x-axis	$[a, b]$ on x-axis	$[c, d]$ on y-axis
Rectangle	Vertical	Vertical	Horizontal
Typical region			
Typical element			

Let's take a look at a couple of additional problems and decide on the best approach to take for solving them.

Example:

Exercise:

Problem:

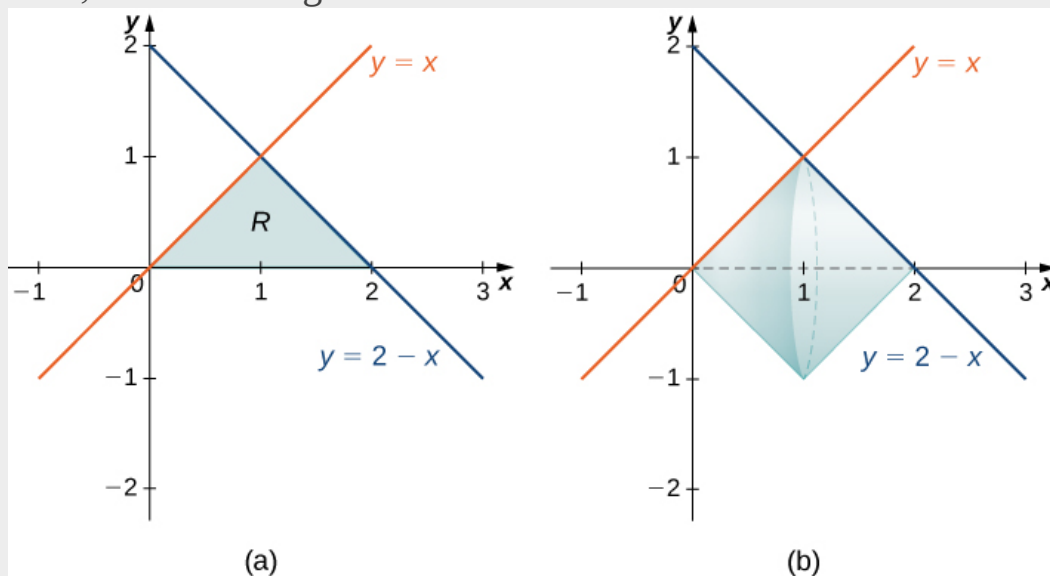
Selecting the Best Method

For each of the following problems, select the best method to find the volume of a solid of revolution generated by revolving the given region around the x -axis, and set up the integral to find the volume (do not evaluate the integral).

- The region bounded by the graphs of $y = x$, $y = 2 - x$, and the x -axis.
- The region bounded by the graphs of $y = 4x - x^2$ and the x -axis.

Solution:

- First, sketch the region and the solid of revolution as shown.



(a) The region R bounded by two lines and the x -axis. (b) The solid of revolution generated by revolving R about the x -axis.

Looking at the region, if we want to integrate with respect to x , we would have to break the integral into two pieces, because we have different functions bounding the region over $[0, 1]$ and $[1, 2]$. In this case, using the disk method, we would have

Equation:

$$V = \int_0^1 (\pi x^2) dx + \int_1^2 (\pi(2 - x)^2) dx.$$

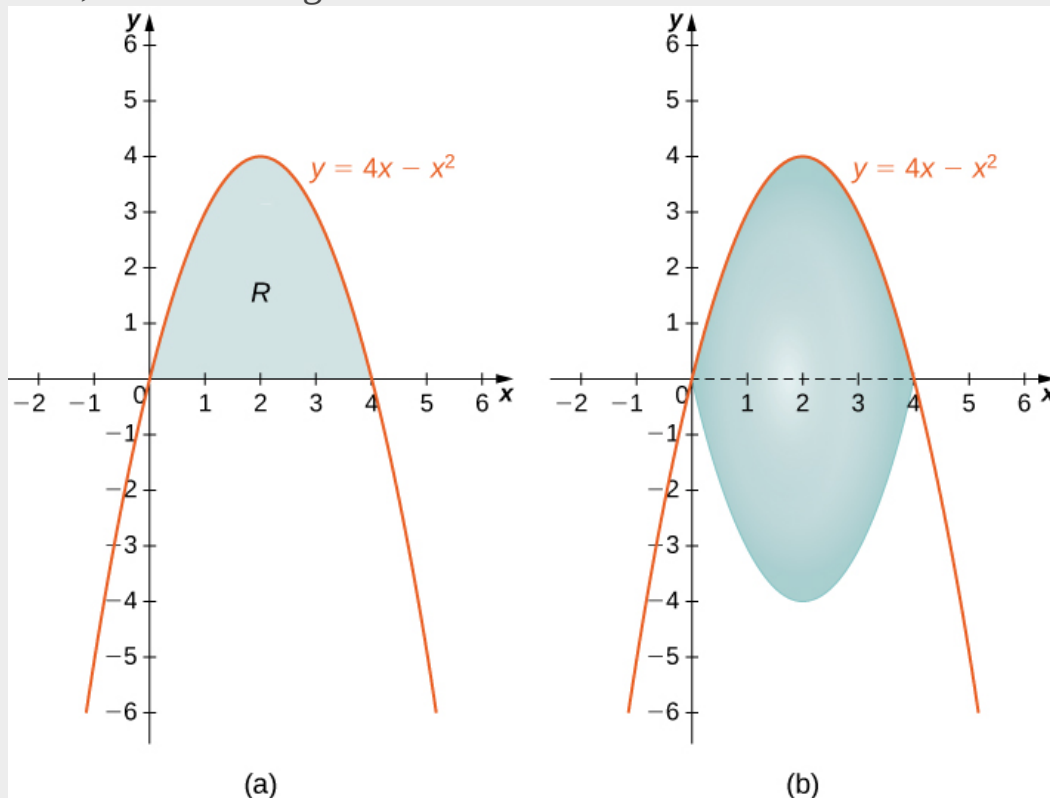
If we used the shell method instead, we would use functions of y to represent the curves, producing

Equation:

$$\begin{aligned} V &= \int_0^1 (2\pi y [(2 - y) - y]) dy \\ &= \int_0^1 (2\pi y [2 - 2y]) dy. \end{aligned}$$

Neither of these integrals is particularly onerous, but since the shell method requires only one integral, and the integrand requires less simplification, we should probably go with the shell method in this case.

- b. First, sketch the region and the solid of revolution as shown.



- (a) The region R between the curve and the x -axis. (b) The solid of revolution generated by revolving R about the

x -axis.

Looking at the region, it would be problematic to define a horizontal rectangle; the region is bounded on the left and right by the same function. Therefore, we can dismiss the method of shells. The solid has no cavity in the middle, so we can use the method of disks. Then

Equation:

$$V = \int_0^4 \pi (4x - x^2)^2 dx.$$

Note:

Exercise:

Problem:

Select the best method to find the volume of a solid of revolution generated by revolving the given region around the x -axis, and set up the integral to find the volume (do not evaluate the integral): the region bounded by the graphs of $y = 2 - x^2$ and $y = x^2$.

Solution:

Use the method of washers; $V = \int_{-1}^1 \pi \left[(2 - x^2)^2 - (x^2)^2 \right] dx$

Hint

Sketch the region and use [\[link\]](#) to decide which integral is easiest to evaluate.

Key Concepts

- The method of cylindrical shells is another method for using a definite integral to calculate the volume of a solid of revolution. This method is sometimes preferable to either the method of disks or the method of washers because we integrate with respect to the other variable. In some cases, one integral is substantially more complicated than the other.
- The geometry of the functions and the difficulty of the integration are the main factors in deciding which integration method to use.

Key Equations

- **Method of Cylindrical Shells**

$$V = \int_a^b (2\pi x f(x)) dx$$

For the following exercises, find the volume generated when the region between the two curves is rotated around the given axis. Use both the shell method and the washer method. Use technology to graph the functions and draw a typical slice by hand.

Exercise:

Problem:

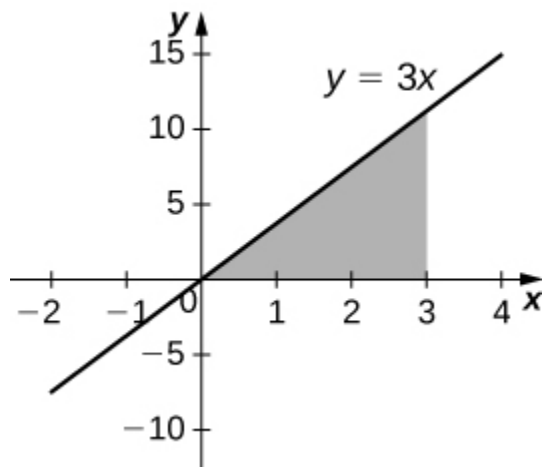
[T] Bounded by the curves $y = 3x$, $x = 0$, and $y = 3$ rotated around the y -axis.

Exercise:

Problem:

[T] Bounded by the curves $y = 3x$, $y = 0$, and $x = 3$ rotated around the y -axis.

Solution:



$54\pi \text{ units}^3$

Exercise:

Problem:

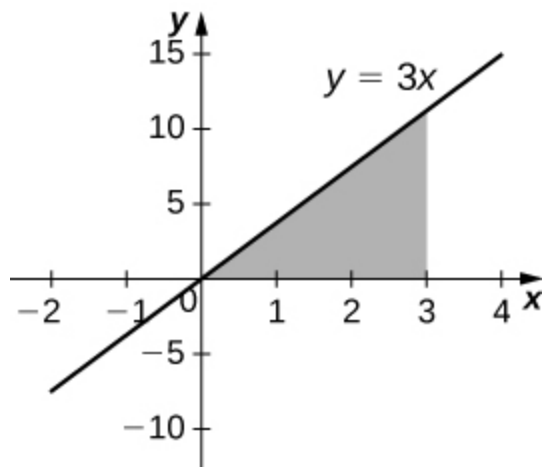
[T] Bounded by the curves $y = 3x$, $y = 0$, and $y = 3$ rotated around the x -axis.

Exercise:

Problem:

[T] Bounded by the curves $y = 3x$, $y = 0$, and $x = 3$ rotated around the x -axis.

Solution:



$81\pi \text{ units}^3$

Exercise:

Problem:

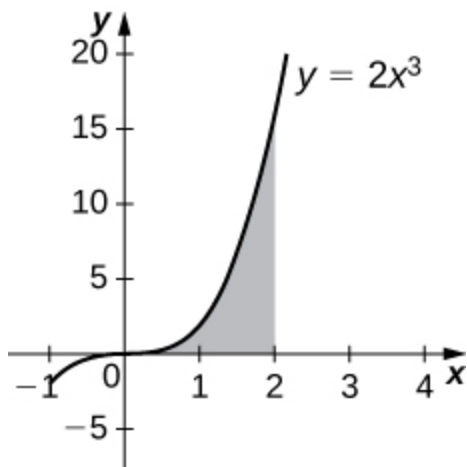
[T] Bounded by the curves $y = 2x^3$, $y = 0$, and $x = 2$ rotated around the y -axis.

Exercise:

Problem:

[T] Bounded by the curves $y = 2x^3$, $y = 0$, and $x = 2$ rotated around the x -axis.

Solution:



$$\frac{512\pi}{7} \text{ units}^3$$

For the following exercises, use shells to find the volumes of the given solids. Note that the rotated regions lie between the curve and the x -axis and are rotated around the y -axis.

Exercise:

Problem: $y = 1 - x^2$, $x = 0$, and $x = 1$

Exercise:

Problem: $y = 5x^3$, $x = 0$, and $x = 1$

Solution:

$$2\pi \text{ units}^3$$

Exercise:

Problem: $y = \frac{1}{x}$, $x = 1$, and $x = 100$

Exercise:

Problem: $y = \sqrt{1 - x^2}$, $x = 0$, and $x = 1$

Solution:

$$\frac{2\pi}{3} \text{ units}^3$$

Exercise:

Problem: $y = \frac{1}{1+x^2}$, $x = 0$, and $x = 3$

Exercise:

Problem: $y = \sin x^2$, $x = 0$, and $x = \sqrt{\pi}$

Solution:

$$2\pi \text{ units}^3$$

Exercise:

Problem: $y = \frac{1}{\sqrt{1-x^2}}$, $x = 0$, and $x = \frac{1}{2}$

Exercise:

Problem: $y = \sqrt{x}$, $x = 0$, and $x = 1$

Solution:

$$\frac{4\pi}{5} \text{ units}^3$$

Exercise:

Problem: $y = (1 + x^2)^3$, $x = 0$, and $x = 1$

Exercise:

Problem: $y = 5x^3 - 2x^4$, $x = 0$, and $x = 2$

Solution:

$$\frac{64\pi}{3} \text{ units}^3$$

For the following exercises, use shells to find the volume generated by rotating the regions between the given curve and $y = 0$ around the x -axis.

Exercise:

Problem: $y = \sqrt{1 - x^2}$, $x = 0$, $x = 1$ and the x -axis

Exercise:

Problem: $y = x^2$, $x = 0$, $x = 2$ and the x -axis

Solution:

$$\frac{32\pi}{5} \text{ units}^3$$

Exercise:

Problem: $y = \frac{x^3}{2}$, $x = 0$, $x = 2$, and the x -axis

Exercise:

Problem: $y = \frac{2}{x^2}$, $x = 1$, $x = 2$, and the x -axis

Solution:

$$\frac{7\pi}{6}$$

Exercise:

Problem: $x = \frac{1}{1+y^2}$, $x = \frac{1}{5}$, and $y = 0$

Exercise:

Problem: $x = \frac{1+y^2}{y}$, $y = 1$, $y = 4$, and the y -axis

Solution:

$$48\pi$$

Exercise:

Problem: $x = \cos y$, $y = 0$, and $y = \pi$

Exercise:

Problem: $x = y^3 - 2y^2$, $x = 0$, $x = 9$, and the y -axis

Solution:

$$\frac{97\pi}{5}$$

Exercise:

Problem: $x = \sqrt{y} + 1$, $x = 1$, $x = 3$, and the x -axis

Exercise:

Problem: $x = \sqrt[3]{27y}$ and $x = \frac{3y}{4}$

Solution:

$$\frac{512\pi}{7}$$

For the following exercises, find the volume generated when the region between the curves is rotated around the given axis.

Exercise:

Problem:

$y = 3 - x$, $y = 0$, $x = 0$, and $x = 2$ rotated around the y -axis.

Exercise:

Problem: $y = x^3$, $x = 0$, and $y = 8$ rotated around the y -axis.

Solution:

$$\frac{64\pi}{5} \text{ units}^3$$

Exercise:

Problem: $y = x^2$, $y = x$, rotated around the y -axis.

Exercise:

Problem: $y = \sqrt{x}$, $y = 0$, and $x = 1$ rotated around the line $x = 2$.

Solution:

$$\frac{28\pi}{15} \text{ units}^3$$

Exercise:

Problem: $y = \frac{1}{4-x}$, $x = 1$, and $x = 2$ rotated around the line $x = 4$.

Exercise:

Problem: $y = \sqrt{x}$ and $y = x^2$ rotated around the y -axis.

Solution:

$$\frac{3\pi}{10} \text{ units}^3$$

Exercise:

Problem: $y = \sqrt{x}$ and $y = x^2$ rotated around the line $x = 2$.

Exercise:

Problem:

$x = y^3$, $x = \frac{1}{y}$, $x = 1$, and $x = 2$ rotated around the x -axis.

Solution:

$$\frac{52\pi}{5} \text{ units}^3$$

Exercise:

Problem: $x = y^2$ and $y = x$ rotated around the line $y = 2$.

Exercise:

Problem: [T] Left of $x = \sin(\pi y)$, right of $y = x$, around the y -axis.

Solution:

$$0.9876 \text{ units}^3$$

For the following exercises, use technology to graph the region. Determine which method you think would be easiest to use to calculate the volume generated when the function is rotated around the specified axis. Then, use your chosen method to find the volume.

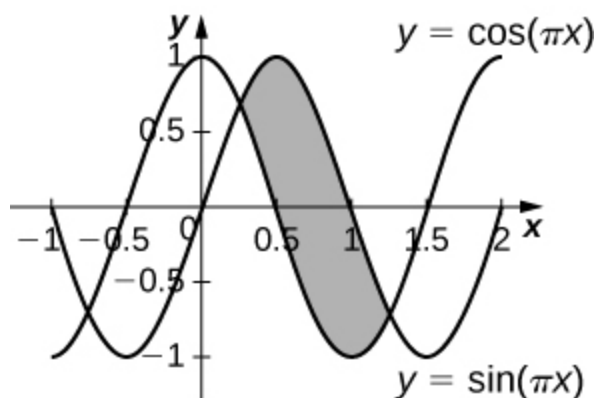
Exercise:

Problem: [T] $y = x^2$ and $y = 4x$ rotated around the y -axis.

Exercise:**Problem:**

[T] $y = \cos(\pi x)$, $y = \sin(\pi x)$, $x = \frac{1}{4}$, and $x = \frac{5}{4}$ rotated around the y -axis.

Solution:



$$3\sqrt{2} \text{ units}^3$$

Exercise:

Problem:

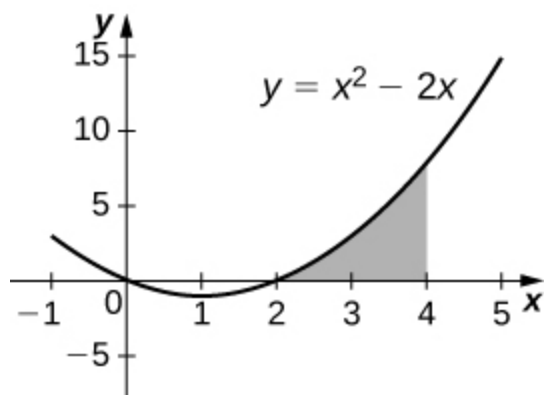
[T] $y = x^2 - 2x$, $x = 2$, and $x = 4$ rotated around the y -axis.

Exercise:

Problem:

[T] $y = x^2 - 2x$, $x = 2$, and $x = 4$ rotated around the x -axis.

Solution:



$$\frac{496\pi}{15} \text{ units}^3$$

Exercise:

Problem:

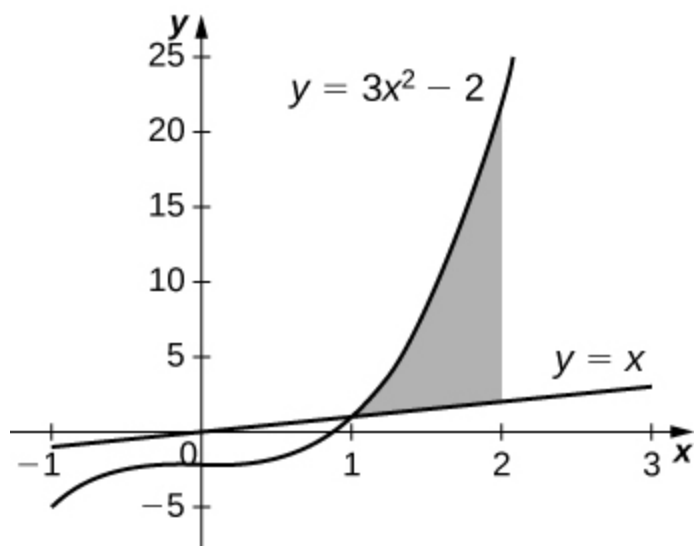
[T] $y = 3x^3 - 2$, $y = x$, and $x = 2$ rotated around the x -axis.

Exercise:

Problem:

[T] $y = 3x^3 - 2$, $y = x$, and $x = 2$ rotated around the y -axis.

Solution:



$$\frac{398\pi}{15} \text{ units}^3$$

Exercise:

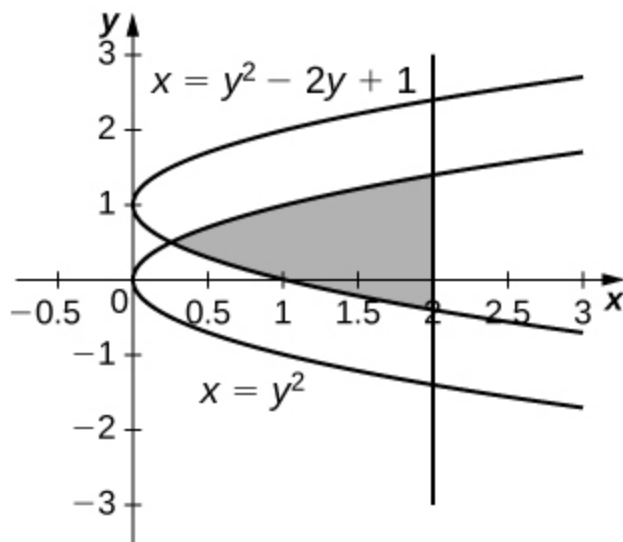
Problem: [T] $x = \sin(\pi y^2)$ and $x = \sqrt{2}y$ rotated around the x -axis.

Exercise:

Problem:

[T] $x = y^2$, $x = y^2 - 2y + 1$, and $x = 2$ rotated around the y -axis.

Solution:



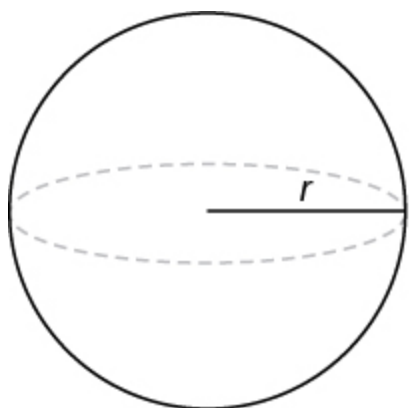
15.9074 units³

For the following exercises, use the method of shells to approximate the volumes of some common objects, which are pictured in accompanying figures.

Exercise:

Problem:

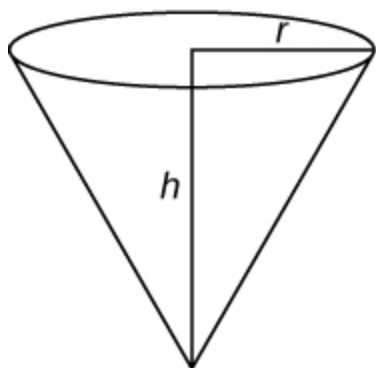
Use the method of shells to find the volume of a sphere of radius r .



Exercise:

Problem:

Use the method of shells to find the volume of a cone with radius r and height h .



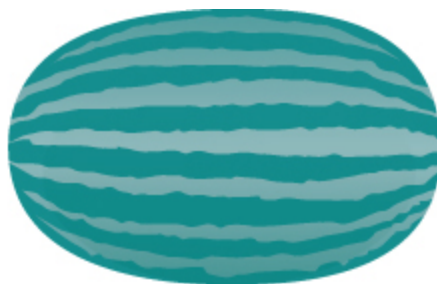
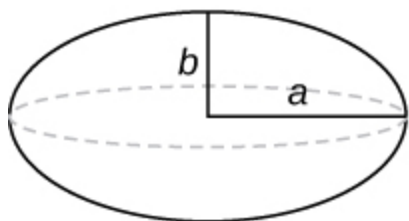
Solution:

$$\frac{1}{3} \pi r^2 h \text{ units}^3$$

Exercise:

Problem:

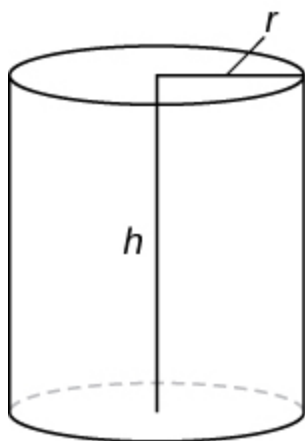
Use the method of shells to find the volume of an ellipsoid $(x^2/a^2) + (y^2/b^2) = 1$ rotated around the x -axis.



Exercise:

Problem:

Use the method of shells to find the volume of a cylinder with radius r and height h .



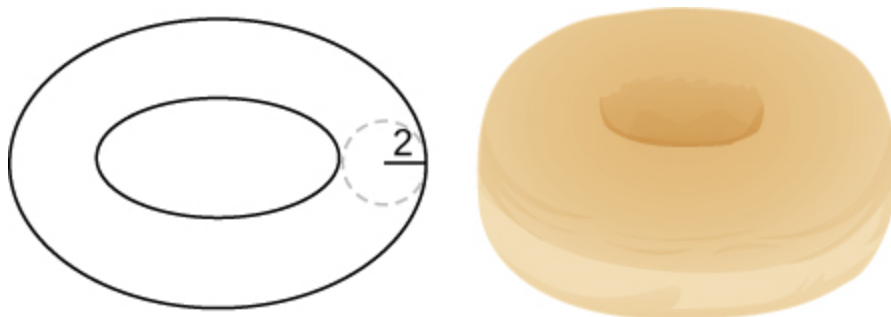
Solution:

$$\pi r^2 h \text{ units}^3$$

Exercise:

Problem:

Use the method of shells to find the volume of the donut created when the circle $x^2 + y^2 = 4$ is rotated around the line $x = 4$.



Exercise:

Problem:

Consider the region enclosed by the graphs of $y = f(x)$, $y = 1 + f(x)$, $x = 0$, $y = 0$, and $x = a > 0$. What is the volume of the solid generated when this region is rotated around the y -axis? Assume that the function is defined over the interval $[0, a]$.

Solution:

$$\pi a^2 \text{ units}^3$$

Exercise:

Problem:

Consider the function $y = f(x)$, which decreases from $f(0) = b$ to $f(1) = 0$. Set up the integrals for determining the volume, using both the shell method and the disk method, of the solid generated when this region, with $x = 0$ and $y = 0$, is rotated around the y -axis. Prove that both methods approximate the same volume. Which method is easier to apply? (*Hint: Since $f(x)$ is one-to-one, there exists an inverse $f^{-1}(y)$.*)

Glossary

method of cylindrical shells

a method of calculating the volume of a solid of revolution by dividing the solid into nested cylindrical shells; this method is different from

the methods of disks or washers in that we integrate with respect to the opposite variable

Moments and Centers of Mass

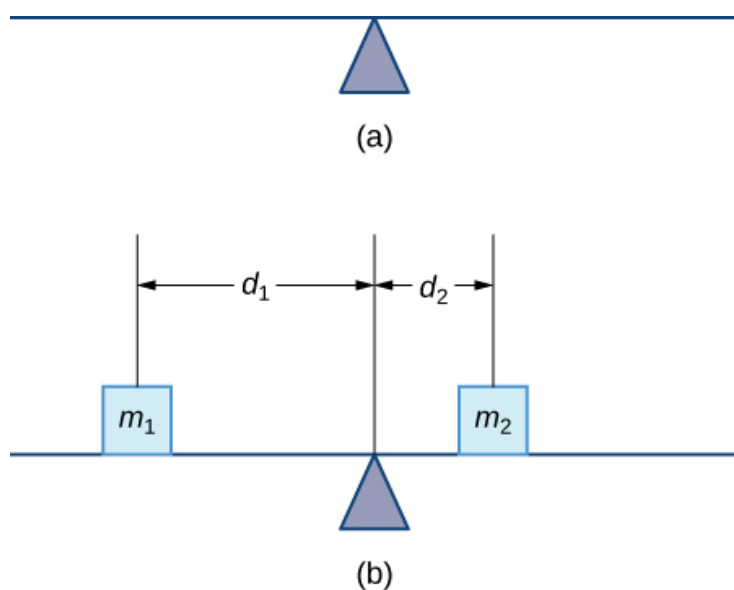
- Find the center of mass of objects distributed along a line.
- Locate the center of mass of a thin plate.
- Use symmetry to help locate the centroid of a thin plate.
- Apply the theorem of Pappus for volume.

In this section, we consider centers of mass (also called *centroids*, under certain conditions) and moments. The basic idea of the center of mass is the notion of a balancing point. Many of us have seen performers who spin plates on the ends of sticks. The performers try to keep several of them spinning without allowing any of them to drop. If we look at a single plate (without spinning it), there is a sweet spot on the plate where it balances perfectly on the stick. If we put the stick anywhere other than that sweet spot, the plate does not balance and it falls to the ground. (That is why performers spin the plates; the spin helps keep the plates from falling even if the stick is not exactly in the right place.) Mathematically, that sweet spot is called the *center of mass of the plate*.

In this section, we first examine these concepts in a one-dimensional context, then expand our development to consider centers of mass of two-dimensional regions and symmetry. Last, we use centroids to find the volume of certain solids by applying the theorem of Pappus.

Center of Mass and Moments

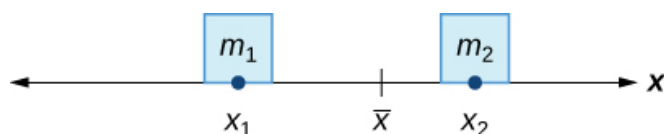
Let's begin by looking at the center of mass in a one-dimensional context. Consider a long, thin wire or rod of negligible mass resting on a fulcrum, as shown in [\[link\]\(a\)](#). Now suppose we place objects having masses m_1 and m_2 at distances d_1 and d_2 from the fulcrum, respectively, as shown in [\[link\]\(b\)](#).



(a) A thin rod rests on a fulcrum. (b) Masses are placed on the rod.

The most common real-life example of a system like this is a playground seesaw, or teeter-totter, with children of different weights sitting at different distances from the center. On a seesaw, if one child sits at each end, the heavier child sinks down and the lighter child is lifted into the air. If the heavier child slides in toward the center, though, the seesaw balances. Applying this concept to the masses on the rod, we note that the masses balance each other if and only if $m_1 d_1 = m_2 d_2$.

In the seesaw example, we balanced the system by moving the masses (children) with respect to the fulcrum. However, we are really interested in systems in which the masses are not allowed to move, and instead we balance the system by moving the fulcrum. Suppose we have two point masses, m_1 and m_2 , located on a number line at points x_1 and x_2 , respectively ([Link](#)). The center of mass, $\bar{x}$, is the point where the fulcrum should be placed to make the system balance.



The center of mass $\bar{x}$ is the balance point of the system.

Thus, we have

Equation:

$$\begin{aligned}
 m_1 |\bar{x} - x_1| &= m_2 |x_2 - \bar{x}| \\
 m_1 (\bar{x} - x_1) &= m_2 (x_2 - \bar{x}) \\
 m_1 \bar{x} - m_1 x_1 &= m_2 x_2 - m_2 \bar{x} \\
 \bar{x} (m_1 + m_2) &= m_1 x_1 + m_2 x_2 \\
 \bar{x} &= \frac{m_1 x_1 + m_2 x_2}{m_1 + m_2}.
 \end{aligned}$$

The expression in the numerator, $m_1 x_1 + m_2 x_2$, is called the *first moment of the system with respect to the origin*. If the context is clear, we often drop the word *first* and just refer to this expression as the **moment** of the system. The expression in the denominator, $m_1 + m_2$, is the total mass of the system. Thus, the **center of mass** of the system is the point at which the total mass of the system could be concentrated without changing the moment.

This idea is not limited just to two point masses. In general, if n masses, $m_1, m_2, \dots, m_n$, are placed on a number line at points $x_1, x_2, \dots, x_n$, respectively, then the center of mass of the system is given by

Equation:

$$\bar{x} = \frac{\sum_{i=1}^n m_i x_i}{\sum_{i=1}^n m_i}.$$

Note:

Center of Mass of Objects on a Line

Let $m_1, m_2, \dots, m_n$ be point masses placed on a number line at points $x_1, x_2, \dots, x_n$,

respectively, and let $m = \sum_{i=1}^n m_i$ denote the total mass of the system. Then, the moment of the system with respect to the origin is given by

Equation:

$$M = \sum_{i=1}^n m_i x_i$$

and the center of mass of the system is given by

Equation:

$$\bar{x} = \frac{M}{m}.$$

We apply this theorem in the following example.

Example:

Exercise:

Problem:

Finding the Center of Mass of Objects along a Line

Suppose four point masses are placed on a number line as follows:

Equation:

$m_1 = 30$ kg, placed at $x_1 = -2$ m

$m_2 = 5$ kg, placed at $x_2 = 3$ m

$m_3 = 10$ kg, placed at $x_3 = 6$ m

$m_4 = 15$ kg, placed at $x_4 = -3$ m.

Find the moment of the system with respect to the origin and find the center of mass of the system.

Solution:

First, we need to calculate the moment of the system:

Equation:

$$\begin{aligned} M &= \sum_{i=1}^4 m_i x_i \\ &= -60 + 15 + 60 - 45 = -30. \end{aligned}$$

Now, to find the center of mass, we need the total mass of the system:

Equation:

$$\begin{aligned} m &= \sum_{i=1}^4 m_i \\ &= 30 + 5 + 10 + 15 = 60 \text{ kg}. \end{aligned}$$

Then we have

Equation:

$$\bar{x} = \frac{M}{m} = \frac{-30}{60} = -\frac{1}{2}.$$

The center of mass is located $1/2$ m to the left of the origin.

Note:

Exercise:

Problem: Suppose four point masses are placed on a number line as follows:

Equation:

$m_1 = 12$ kg, placed at $x_1 = -4$ m

$m_2 = 12$ kg, placed at $x_2 = 4$ m

$m_3 = 30$ kg, placed at $x_3 = 2$ m

$m_4 = 6$ kg, placed at $x_4 = -6$ m.

Find the moment of the system with respect to the origin and find the center of mass of the system.

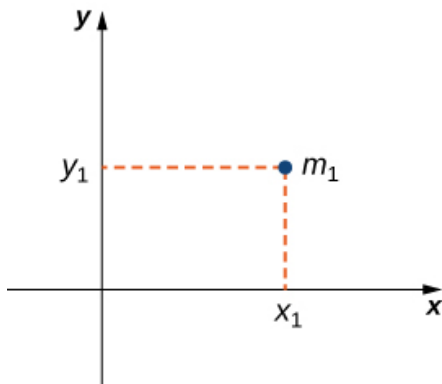
Solution:

$$M = 24, \bar{x} = \frac{2}{5} \text{ m}$$

Hint

Use the process from the previous example.

We can generalize this concept to find the center of mass of a system of point masses in a plane. Let m_1 be a point mass located at point (x_1, y_1) in the plane. Then the moment M_x of the mass with respect to the x -axis is given by $M_x = m_1 y_1$. Similarly, the moment M_y with respect to the y -axis is given by $M_y = m_1 x_1$. Notice that the x -coordinate of the point is used to calculate the moment with respect to the y -axis, and vice versa. The reason is that the x -coordinate gives the distance from the point mass to the y -axis, and the y -coordinate gives the distance to the x -axis (see the following figure).



Point mass m_1 is located at point (x_1, y_1) in the plane.

If we have several point masses in the xy -plane, we can use the moments with respect to the x - and y -axes to calculate the x - and y -coordinates of the center of mass of the system.

Note:

Center of Mass of Objects in a Plane

Let $m_1, m_2, \dots, m_n$ be point masses located in the xy -plane at points $(x_1, y_1), (x_2, y_2), \dots, (x_n, y_n)$, respectively, and let $m = \sum_{i=1}^n m_i$ denote the total mass of the system. Then the moments M_x and M_y of the system with respect to the x - and y -axes, respectively, are given by

Equation:

$$M_x = \sum_{i=1}^n m_i y_i \quad \text{and} \quad M_y = \sum_{i=1}^n m_i x_i.$$

Also, the coordinates of the center of mass $(\bar{x}, \bar{y})$ of the system are

Equation:

$$\bar{x} = \frac{M_y}{m} \quad \text{and} \quad \bar{y} = \frac{M_x}{m}.$$

The next example demonstrates how to apply this theorem.

Example:

Exercise:

Problem:

Finding the Center of Mass of Objects in a Plane

Suppose three point masses are placed in the xy -plane as follows (assume coordinates are given in meters):

Equation:

$$\begin{aligned} m_1 &= 2 \text{ kg, placed at } (-1, 3), \\ m_2 &= 6 \text{ kg, placed at } (1, 1), \\ m_3 &= 4 \text{ kg, placed at } (2, -2). \end{aligned}$$

Find the center of mass of the system.

Solution:

First we calculate the total mass of the system:

Equation:

$$m = \sum_{i=1}^3 m_i = 2 + 6 + 4 = 12 \text{ kg.}$$

Next we find the moments with respect to the x- and y-axes:

Equation:

$$M_y = \sum_{i=1}^3 m_i x_i = -2 + 6 + 8 = 12,$$

$$M_x = \sum_{i=1}^3 m_i y_i = 6 + 6 - 8 = 4.$$

Then we have

Equation:

$$\bar{x} = \frac{M_y}{m} = \frac{12}{12} = 1 \text{ and } \bar{y} = \frac{M_x}{m} = \frac{4}{12} = \frac{1}{3}.$$

The center of mass of the system is $(1, 1/3)$, in meters.

Note:

Exercise:

Problem:

Suppose three point masses are placed on a number line as follows (assume coordinates are given in meters):

Equation:

$$m_1 = 5 \text{ kg, placed at } (-2, -3),$$

$$m_2 = 3 \text{ kg, placed at } (2, 3),$$

$$m_3 = 2 \text{ kg, placed at } (-3, -2).$$

Find the center of mass of the system.

Solution:

$$(-1, -1) \text{ m}$$

Hint

Use the process from the previous example.

Center of Mass of Thin Plates

So far we have looked at systems of point masses on a line and in a plane. Now, instead of having the mass of a system concentrated at discrete points, we want to look at systems in which the mass of the system is distributed continuously across a thin sheet of material. For our purposes, we assume the sheet is thin enough that it can be treated as if it is two-dimensional. Such a sheet is called a **lamina**. Next we develop techniques to find the center of mass of a lamina. In this section, we also assume the density of the lamina is constant.

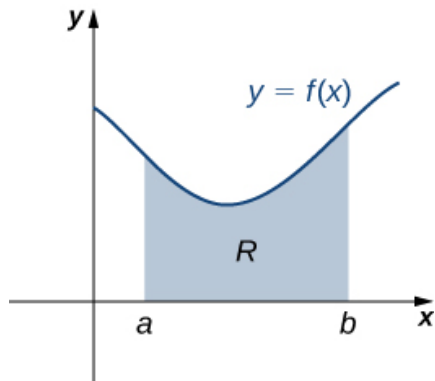
Laminas are often represented by a two-dimensional region in a plane. The geometric center of such a region is called its **centroid**. Since we have assumed the density of the lamina is constant, the center of mass of the lamina depends only on the shape of the corresponding region in the plane; it does not depend on the density. In this case, the center of mass of the lamina corresponds to the centroid of the delineated region in the plane. As with systems of point masses, we need to find the total mass of the lamina, as well as the moments of the lamina with respect to the x - and y -axes.

We first consider a lamina in the shape of a rectangle. Recall that the center of mass of a lamina is the point where the lamina balances. For a rectangle, that point is both the horizontal and vertical center of the rectangle. Based on this understanding, it is clear that the center of mass of a rectangular lamina is the point where the diagonals intersect, which is a result of the **symmetry principle**, and it is stated here without proof.

Note:**The Symmetry Principle**

If a region R is symmetric about a line l , then the centroid of R lies on l .

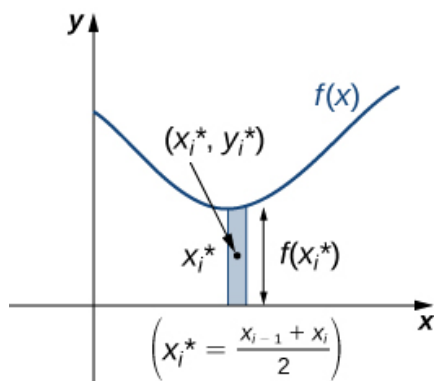
Let's turn to more general laminas. Suppose we have a lamina bounded above by the graph of a continuous function $f(x)$, below by the x -axis, and on the left and right by the lines $x = a$ and $x = b$, respectively, as shown in the following figure.



A region in the plane
representing a lamina.

As with systems of point masses, to find the center of mass of the lamina, we need to find the total mass of the lamina, as well as the moments of the lamina with respect to the x - and y -axes. As we have done many times before, we approximate these quantities by partitioning the interval $[a, b]$ and constructing rectangles.

For $i = 0, 1, 2, \dots, n$, let $P = \{x_i\}$ be a regular partition of $[a, b]$. Recall that we can choose any point within the interval $[x_{i-1}, x_i]$ as our x_i^* . In this case, we want x_i^* to be the x -coordinate of the centroid of our rectangles. Thus, for $i = 1, 2, \dots, n$, we select $x_i^* \in [x_{i-1}, x_i]$ such that x_i^* is the midpoint of the interval. That is, $x_i^* = (x_{i-1} + x_i)/2$. Now, for $i = 1, 2, \dots, n$, construct a rectangle of height $f(x_i^*)$ on $[x_{i-1}, x_i]$. The center of mass of this rectangle is $(x_i^*, (f(x_i^*))/2)$, as shown in the following figure.



A representative rectangle
of the lamina.

Next, we need to find the total mass of the rectangle. Let ρ represent the density of the lamina (note that ρ is a constant). In this case, ρ is expressed in terms of mass per unit area. Thus, to find the total mass of the rectangle, we multiply the area of the rectangle by ρ . Then, the mass of the rectangle is given by $\rho f(x_i^*) \Delta x$.

To get the approximate mass of the lamina, we add the masses of all the rectangles to get
Equation:

$$m \approx \sum_{i=1}^n \rho f(x_i^*) \Delta x.$$

This is a Riemann sum. Taking the limit as $n \rightarrow \infty$ gives the exact mass of the lamina:
Equation:

$$m = \lim_{n \rightarrow \infty} \sum_{i=1}^n \rho f(x_i^*) \Delta x = \rho \int_a^b f(x) dx.$$

Next, we calculate the moment of the lamina with respect to the x -axis. Returning to the representative rectangle, recall its center of mass is $(x_i^*, (f(x_i^*)/2))$. Recall also that treating the rectangle as if it is a point mass located at the center of mass does not change the moment. Thus, the moment of the rectangle with respect to the x -axis is given by the mass of the rectangle, $\rho f(x_i^*) \Delta x$, multiplied by the distance from the center of mass to the x -axis: $(f(x_i^*)/2)$. Therefore, the moment with respect to the x -axis of the rectangle is $\rho \left([f(x_i^*)]^2 / 2 \right) \Delta x$. Adding the moments of the rectangles and taking the limit of the resulting Riemann sum, we see that the moment of the lamina with respect to the x -axis is
Equation:

$$M_x = \lim_{n \rightarrow \infty} \sum_{i=1}^n \rho \frac{[f(x_i^*)]^2}{2} \Delta x = \rho \int_a^b \frac{[f(x)]^2}{2} dx.$$

We derive the moment with respect to the y -axis similarly, noting that the distance from the center of mass of the rectangle to the y -axis is x_i^* . Then the moment of the lamina with respect to the y -axis is given by

Equation:

$$M_y = \lim_{n \rightarrow \infty} \sum_{i=1}^n \rho x_i^* f(x_i^*) \Delta x = \rho \int_a^b x f(x) dx.$$

We find the coordinates of the center of mass by dividing the moments by the total mass to give $\bar{x} = M_y/m$ and $\bar{y} = M_x/m$. If we look closely at the expressions for M_x , M_y , and m , we notice that the constant ρ cancels out when $\bar{x}$ and $\bar{y}$ are calculated.

We summarize these findings in the following theorem.

Note:

Center of Mass of a Thin Plate in the xy -Plane

Let R denote a region bounded above by the graph of a continuous function $f(x)$, below by the x -axis, and on the left and right by the lines $x = a$ and $x = b$, respectively. Let ρ denote the density of the associated lamina. Then we can make the following statements:

- i. The mass of the lamina is

Equation:

$$m = \rho \int_a^b f(x) dx.$$

- ii. The moments M_x and M_y of the lamina with respect to the x - and y -axes, respectively, are

Equation:

$$M_x = \rho \int_a^b \frac{[f(x)]^2}{2} dx \text{ and } M_y = \rho \int_a^b x f(x) dx.$$

- iii. The coordinates of the center of mass $(\bar{x}, \bar{y})$ are

Equation:

$$\bar{x} = \frac{M_y}{m} \text{ and } \bar{y} = \frac{M_x}{m}.$$

In the next example, we use this theorem to find the center of mass of a lamina.

Example:

Exercise:

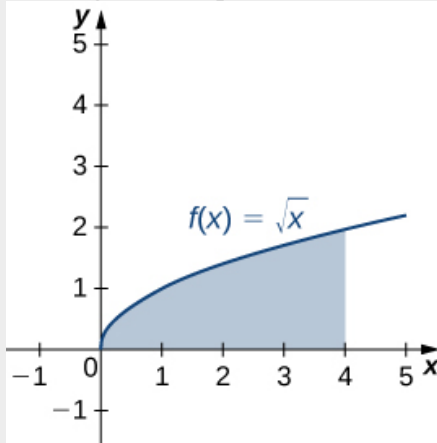
Problem:

Finding the Center of Mass of a Lamina

Let R be the region bounded above by the graph of the function $f(x) = \sqrt{x}$ and below by the x -axis over the interval $[0, 4]$. Find the centroid of the region.

Solution:

The region is depicted in the following figure.



Finding the center of mass
of a lamina.

Since we are only asked for the centroid of the region, rather than the mass or moments of the associated lamina, we know the density constant ρ cancels out of the calculations eventually. Therefore, for the sake of convenience, let's assume $\rho = 1$.

First, we need to calculate the total mass:

Equation:

$$\begin{aligned} m &= \rho \int_a^b f(x) dx = \int_0^4 \sqrt{x} dx \\ &= \frac{2}{3} x^{3/2} \Big|_0^4 = \frac{2}{3} [8 - 0] = \frac{16}{3}. \end{aligned}$$

Next, we compute the moments:

Equation:

$$\begin{aligned}
 M_x &= \rho \int_a^b \frac{[f(x)]^2}{2} dx \\
 &= \int_0^4 \frac{x}{2} dx = \frac{1}{4} x^2 \Big|_0^4 = 4
 \end{aligned}$$

and

Equation:

$$\begin{aligned}
 M_y &= \rho \int_a^b x f(x) dx \\
 &= \int_0^4 x \sqrt{x} dx = \int_0^4 x^{3/2} dx \\
 &= \frac{2}{5} x^{5/2} \Big|_0^4 = \frac{2}{5} [32 - 0] = \frac{64}{5}.
 \end{aligned}$$

Thus, we have

Equation:

$$\bar{x} = \frac{M_y}{m} = \frac{64/5}{16/3} = \frac{64}{5} \cdot \frac{3}{16} = \frac{12}{5} \text{ and } \bar{y} = \frac{M_x}{m} = \frac{4}{16/3} = 4 \cdot \frac{3}{16} = \frac{3}{4}.$$

The centroid of the region is $(12/5, 3/4)$.

Note:

Exercise:

Problem:

Let R be the region bounded above by the graph of the function $f(x) = x^2$ and below by the x -axis over the interval $[0, 2]$. Find the centroid of the region.

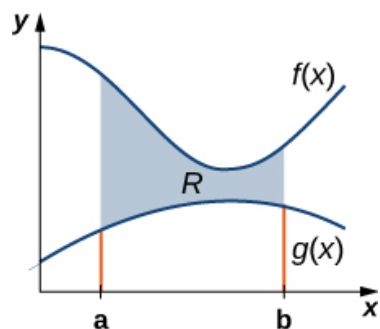
Solution:

The centroid of the region is $(3/2, 6/5)$.

Hint

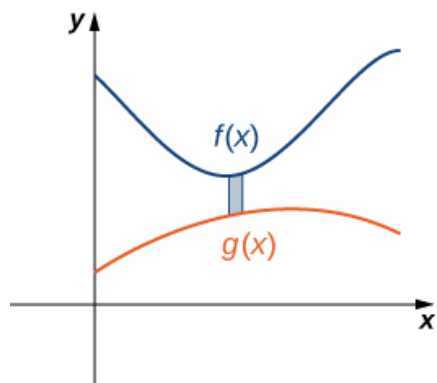
Use the process from the previous example.

We can adapt this approach to find centroids of more complex regions as well. Suppose our region is bounded above by the graph of a continuous function $f(x)$, as before, but now, instead of having the lower bound for the region be the x -axis, suppose the region is bounded below by the graph of a second continuous function, $g(x)$, as shown in the following figure.



A region between two functions.

Again, we partition the interval $[a, b]$ and construct rectangles. A representative rectangle is shown in the following figure.



A representative rectangle of the region between two functions.

Note that the centroid of this rectangle is $(x_i^*, (f(x_i^*) + g(x_i^*))/2)$. We won't go through all the details of the Riemann sum development, but let's look at some of the key steps. In the

development of the formulas for the mass of the lamina and the moment with respect to the y-axis, the height of each rectangle is given by $f(x_i^*) - g(x_i^*)$, which leads to the expression $f(x) - g(x)$ in the integrands.

In the development of the formula for the moment with respect to the x-axis, the moment of each rectangle is found by multiplying the area of the rectangle, $\rho [f(x_i^*) - g(x_i^*)] \Delta x$, by the distance of the centroid from the x-axis, $(f(x_i^*) + g(x_i^*)) / 2$, which gives

$\rho (1/2) \left\{ [f(x_i^*)]^2 - [g(x_i^*)]^2 \right\} \Delta x$. Summarizing these findings, we arrive at the following theorem.

Note:

Center of Mass of a Lamina Bounded by Two Functions

Let R denote a region bounded above by the graph of a continuous function $f(x)$, below by the graph of the continuous function $g(x)$, and on the left and right by the lines $x = a$ and $x = b$, respectively. Let ρ denote the density of the associated lamina. Then we can make the following statements:

- i. The mass of the lamina is

Equation:

$$m = \rho \int_a^b [f(x) - g(x)] dx.$$

- ii. The moments M_x and M_y of the lamina with respect to the x- and y-axes, respectively, are

Equation:

$$M_x = \rho \int_a^b \frac{1}{2} ([f(x)]^2 - [g(x)]^2) dx \text{ and } M_y = \rho \int_a^b x [f(x) - g(x)] dx.$$

- iii. The coordinates of the center of mass $(\bar{x}, \bar{y})$ are

Equation:

$$\bar{x} = \frac{M_y}{m} \text{ and } \bar{y} = \frac{M_x}{m}.$$

We illustrate this theorem in the following example.

Example:

Exercise:

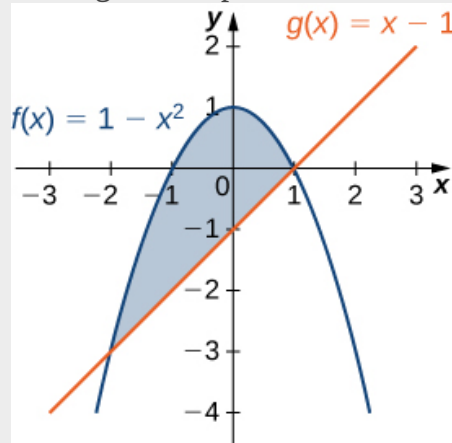
Problem:

Finding the Centroid of a Region Bounded by Two Functions

Let R be the region bounded above by the graph of the function $f(x) = 1 - x^2$ and below by the graph of the function $g(x) = x - 1$. Find the centroid of the region.

Solution:

The region is depicted in the following figure.



Finding the centroid of a region between two curves.

The graphs of the functions intersect at $(-2, -3)$ and $(1, 0)$, so we integrate from -2 to 1 . Once again, for the sake of convenience, assume $\rho = 1$.

First, we need to calculate the total mass:

Equation:

$$\begin{aligned} m &= \rho \int_a^b [f(x) - g(x)] dx \\ &= \int_{-2}^1 [1 - x^2 - (x - 1)] dx = \int_{-2}^1 (2 - x^2 - x) dx \\ &= \left[2x - \frac{1}{3}x^3 - \frac{1}{2}x^2 \right]_{-2}^1 = \left[2 - \frac{1}{3} - \frac{1}{2} \right] - \left[-4 + \frac{8}{3} - 2 \right] = \frac{9}{2}. \end{aligned}$$

Next, we compute the moments:

Equation:

$$\begin{aligned}M_x &= \rho \int_a^b \frac{1}{2} \left([f(x)]^2 - [g(x)]^2 \right) dx \\&= \frac{1}{2} \int_{-2}^1 \left((1 - x^2)^2 - (x - 1)^2 \right) dx = \frac{1}{2} \int_{-2}^1 (x^4 - 3x^2 + 2x) dx \\&= \frac{1}{2} \left[\frac{x^5}{5} - x^3 + x^2 \right] \Big|_{-2}^1 = -\frac{27}{10}\end{aligned}$$

and

Equation:

$$\begin{aligned}M_y &= \rho \int_a^b x [f(x) - g(x)] dx \\&= \int_{-2}^1 x [(1 - x^2) - (x - 1)] dx = \int_{-2}^1 x [2 - x^2 - x] dx \\&= \int_{-2}^1 (2x - x^4 - x^2) dx \\&= \left[x^2 - \frac{x^5}{5} - \frac{x^3}{3} \right] \Big|_{-2}^1 = -\frac{9}{4}.\end{aligned}$$

Therefore, we have

Equation:

$$\bar{x} = \frac{M_y}{m} = -\frac{9}{4} \cdot \frac{2}{9} = -\frac{1}{2} \text{ and } \bar{y} = \frac{M_x}{m} = -\frac{27}{10} \cdot \frac{2}{9} = -\frac{3}{5}.$$

The centroid of the region is $(- (1/2), - (3/5))$.

Note:

Exercise:

Problem:

Let R be the region bounded above by the graph of the function $f(x) = 6 - x^2$ and below by the graph of the function $g(x) = 3 - 2x$. Find the centroid of the region.

Solution:

The centroid of the region is $(1, 13/5)$.

Hint

Use the process from the previous example.

The Symmetry Principle

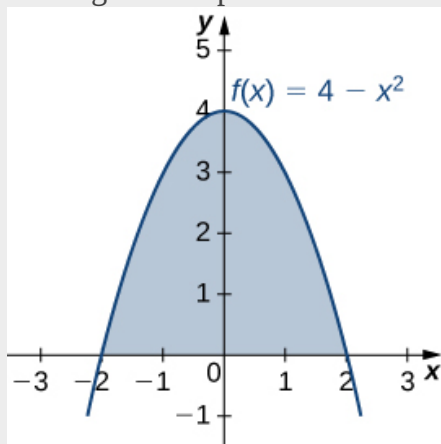
We stated the symmetry principle earlier, when we were looking at the centroid of a rectangle. The symmetry principle can be a great help when finding centroids of regions that are symmetric. Consider the following example.

Example:**Exercise:****Problem:****Finding the Centroid of a Symmetric Region**

Let R be the region bounded above by the graph of the function $f(x) = 4 - x^2$ and below by the x -axis. Find the centroid of the region.

Solution:

The region is depicted in the following figure.



We can use the symmetry principle to help find the centroid of a symmetric region.

The region is symmetric with respect to the y -axis. Therefore, the x -coordinate of the centroid is zero. We need only calculate $\bar{y}$. Once again, for the sake of convenience, assume $\rho = 1$.

First, we calculate the total mass:

Equation:

$$\begin{aligned} m &= \rho \int_a^b f(x) dx \\ &= \int_{-2}^2 (4 - x^2) dx \\ &= \left[4x - \frac{x^3}{3} \right]_{-2}^2 = \frac{32}{3}. \end{aligned}$$

Next, we calculate the moments. We only need M_x :

Equation:

$$\begin{aligned} M_x &= \rho \int_a^b \frac{[f(x)]^2}{2} dx \\ &= \frac{1}{2} \int_{-2}^2 [4 - x^2]^2 dx = \frac{1}{2} \int_{-2}^2 (16 - 8x^2 + x^4) dx \\ &= \frac{1}{2} \left[\frac{x^5}{5} - \frac{8x^3}{3} + 16x \right]_{-2}^2 = \frac{256}{15}. \end{aligned}$$

Then we have

Equation:

$$\bar{y} = \frac{M_x}{m} = \frac{256}{15} \cdot \frac{3}{32} = \frac{8}{5}.$$

The centroid of the region is $(0, 8/5)$.

Note:

Exercise:

Problem:

Let R be the region bounded above by the graph of the function $f(x) = 1 - x^2$ and below by x -axis. Find the centroid of the region.

Solution:

The centroid of the region is $(0, 2/5)$.

Hint

Use the process from the previous example.

Note:**The Grand Canyon Skywalk**

The Grand Canyon Skywalk opened to the public on March 28, 2007. This engineering marvel is a horseshoe-shaped observation platform suspended 4000 ft above the Colorado River on the West Rim of the Grand Canyon. Its crystal-clear glass floor allows stunning views of the canyon below (see the following figure).

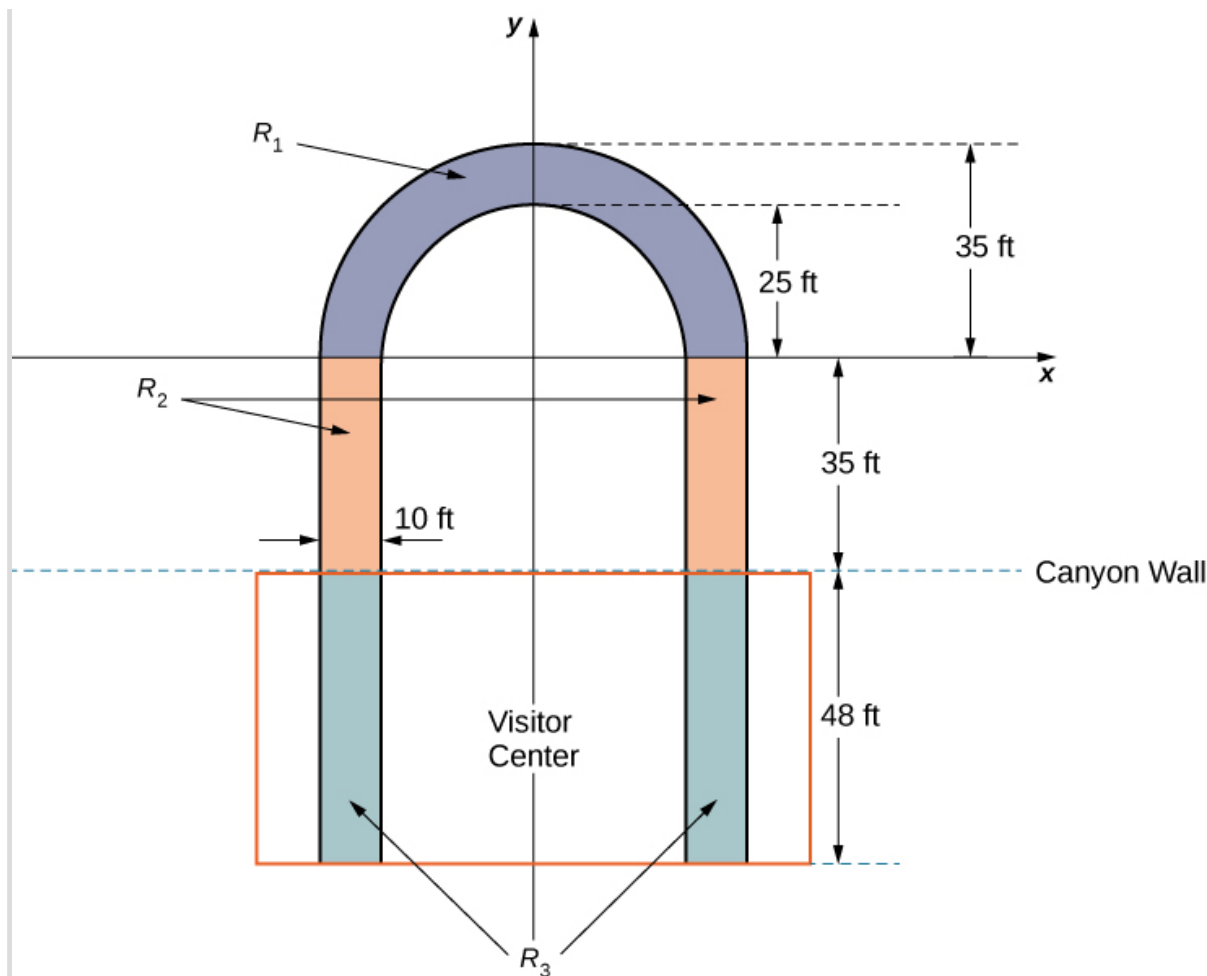


The Grand Canyon Skywalk offers magnificent views of the canyon. (credit: 10da_ralta, Wikimedia Commons)

The Skywalk is a cantilever design, meaning that the observation platform extends over the rim of the canyon, with no visible means of support below it. Despite the lack of visible support posts or struts, cantilever structures are engineered to be very stable and the Skywalk is no exception. The observation platform is attached firmly to support posts that extend 46 ft down into bedrock. The structure was built to withstand 100-mph winds and an 8.0-magnitude earthquake within 50 mi, and is capable of supporting more than 70,000,000 lb. One factor affecting the stability of the Skywalk is the center of gravity of the structure. We are going to calculate the center of gravity of the Skywalk, and examine how the center of gravity changes when tourists walk out onto the observation platform.

The observation platform is U-shaped. The legs of the U are 10 ft wide and begin on land, under the visitors' center, 48 ft from the edge of the canyon. The platform extends 70 ft over the edge of the canyon.

To calculate the center of mass of the structure, we treat it as a lamina and use a two-dimensional region in the xy -plane to represent the platform. We begin by dividing the region into three subregions so we can consider each subregion separately. The first region, denoted R_1 , consists of the curved part of the U. We model R_1 as a semicircular annulus, with inner radius 25 ft and outer radius 35 ft, centered at the origin (see the following figure).



We model the Skywalk with three sub-regions.

The legs of the platform, extending 35 ft between R_1 and the canyon wall, comprise the second sub-region, R_2 . Last, the ends of the legs, which extend 48 ft under the visitor center, comprise the third sub-region, R_3 . Assume the density of the lamina is constant and assume the total weight of the platform is 1,200,000 lb (not including the weight of the visitor center; we will consider that later). Use $g = 32 \text{ ft/sec}^2$.

1. Compute the area of each of the three sub-regions. Note that the areas of regions R_2 and R_3 should include the areas of the legs only, not the open space between them. Round answers to the nearest square foot.
2. Determine the mass associated with each of the three sub-regions.
3. Calculate the center of mass of each of the three sub-regions.
4. Now, treat each of the three sub-regions as a point mass located at the center of mass of the corresponding sub-region. Using this representation, calculate the center of mass of the entire platform.

5. Assume the visitor center weighs 2,200,000 lb, with a center of mass corresponding to the center of mass of R_3 . Treating the visitor center as a point mass, recalculate the center of mass of the system. How does the center of mass change?
6. Although the Skywalk was built to limit the number of people on the observation platform to 120, the platform is capable of supporting up to 800 people weighing 200 lb each. If all 800 people were allowed on the platform, and all of them went to the farthest end of the platform, how would the center of gravity of the system be affected? (Include the visitor center in the calculations and represent the people by a point mass located at the farthest edge of the platform, 70 ft from the canyon wall.)

Theorem of Pappus

This section ends with a discussion of the **theorem of Pappus for volume**, which allows us to find the volume of particular kinds of solids by using the centroid. (There is also a theorem of Pappus for surface area, but it is much less useful than the theorem for volume.)

Note:

Theorem of Pappus for Volume

Let R be a region in the plane and let l be a line in the plane that does not intersect R . Then the volume of the solid of revolution formed by revolving R around l is equal to the area of R multiplied by the distance d traveled by the centroid of R .

Proof

We can prove the case when the region is bounded above by the graph of a function $f(x)$ and below by the graph of a function $g(x)$ over an interval $[a, b]$, and for which the axis of

revolution is the y -axis. In this case, the area of the region is $A = \int_a^b [f(x) - g(x)] dx$.

Since the axis of rotation is the y -axis, the distance traveled by the centroid of the region depends only on the x -coordinate of the centroid, $\bar{x}$, which is

Equation:

$$\bar{x} = \frac{M_y}{m},$$

where

Equation:

$$m = \rho \int_a^b [f(x) - g(x)] dx \text{ and } M_y = \rho \int_a^b x [f(x) - g(x)] dx.$$

Then,

Equation:

$$d = 2\pi \frac{\rho \int_a^b x [f(x) - g(x)] dx}{\rho \int_a^b [f(x) - g(x)] dx}$$

and thus

Equation:

$$d \cdot A = 2\pi \int_a^b x [f(x) - g(x)] dx.$$

However, using the method of cylindrical shells, we have

Equation:

$$V = 2\pi \int_a^b x [f(x) - g(x)] dx.$$

So,

Equation:

$$V = d \cdot A$$

and the proof is complete.

□

Example:

Exercise:

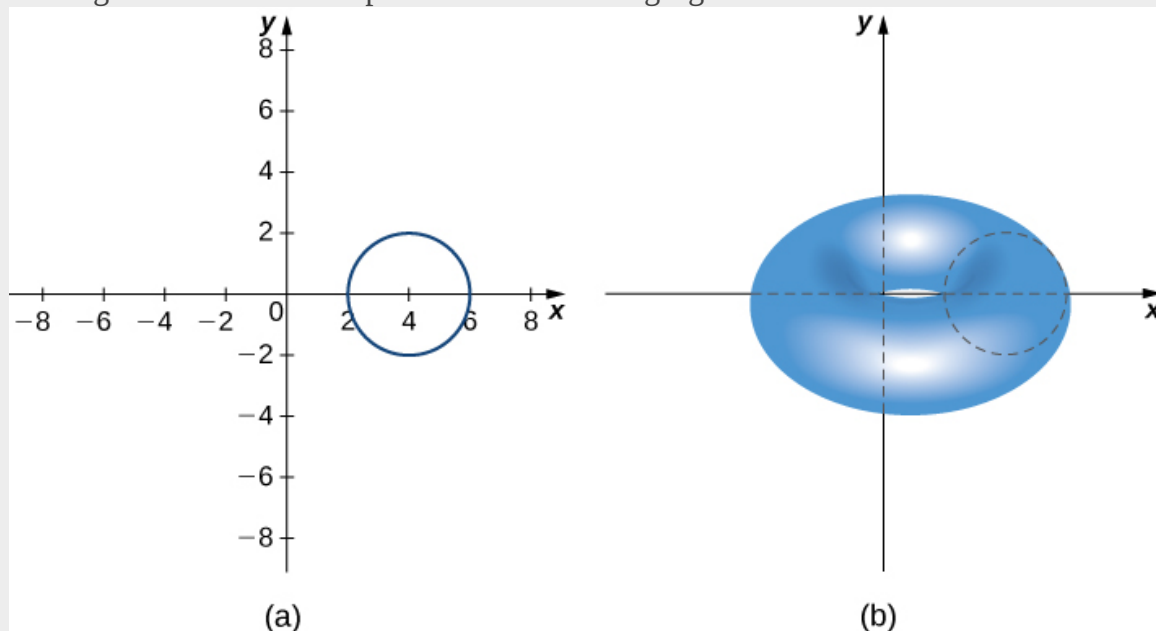
Problem:

Using the Theorem of Pappus for Volume

Let R be a circle of radius 2 centered at $(4, 0)$. Use the theorem of Pappus for volume to find the volume of the torus generated by revolving R around the y -axis.

Solution:

The region and torus are depicted in the following figure.



Determining the volume of a torus by using the theorem of Pappus. (a) A circular region R in the plane; (b) the torus generated by revolving R about the y -axis.

The region R is a circle of radius 2, so the area of R is $A = 4\pi$ units². By the symmetry principle, the centroid of R is the center of the circle. The centroid travels around the y -axis in a circular path of radius 4, so the centroid travels $d = 8\pi$ units. Then, the volume of the torus is $A \cdot d = 32\pi^2$ units³.

Note:

Exercise:

Problem:

Let R be a circle of radius 1 centered at $(3, 0)$. Use the theorem of Pappus for volume to find the volume of the torus generated by revolving R around the y -axis.

Solution:

$$6\pi^2 \text{ units}^3$$

Hint

Use the process from the previous example.

Key Concepts

- Mathematically, the center of mass of a system is the point at which the total mass of the system could be concentrated without changing the moment. Loosely speaking, the center of mass can be thought of as the balancing point of the system.
- For point masses distributed along a number line, the moment of the system with respect to the origin is $M = \sum_{i=1}^n m_i x_i$. For point masses distributed in a plane, the moments of the system with respect to the x- and y-axes, respectively, are $M_x = \sum_{i=1}^n m_i y_i$ and $M_y = \sum_{i=1}^n m_i x_i$, respectively.
- For a lamina bounded above by a function $f(x)$, the moments of the system with respect to the x- and y-axes, respectively, are $M_x = \rho \int_a^b \frac{[f(x)]^2}{2} dx$ and $M_y = \rho \int_a^b x f(x) dx$.
- The x- and y-coordinates of the center of mass can be found by dividing the moments around the y-axis and around the x-axis, respectively, by the total mass. The symmetry principle says that if a region is symmetric with respect to a line, then the centroid of the region lies on the line.
- The theorem of Pappus for volume says that if a region is revolved around an external axis, the volume of the resulting solid is equal to the area of the region multiplied by the distance traveled by the centroid of the region.

Key Equations

- Mass of a lamina**

$$m = \rho \int_a^b f(x) dx$$

- Moments of a lamina**

$$M_x = \rho \int_a^b \frac{[f(x)]^2}{2} dx \text{ and } M_y = \rho \int_a^b x f(x) dx$$

- Center of mass of a lamina**

$$\bar{x} = \frac{M_y}{m} \text{ and } \bar{y} = \frac{M_x}{m}$$

For the following exercises, calculate the center of mass for the collection of masses given.

Exercise:

Problem: $m_1 = 2$ at $x_1 = 1$ and $m_2 = 4$ at $x_2 = 2$

Exercise:

Problem: $m_1 = 1$ at $x_1 = -1$ and $m_2 = 3$ at $x_2 = 2$

Solution:

$$\frac{5}{4}$$

Exercise:

Problem: $m = 3$ at $x = 0, 1, 2, 6$

Exercise:

Problem: Unit masses at $(x, y) = (1, 0), (0, 1), (1, 1)$

Solution:

$$\left(\frac{2}{3}, \frac{2}{3}\right)$$

Exercise:

Problem: $m_1 = 1$ at $(1, 0)$ and $m_2 = 4$ at $(0, 1)$

Exercise:

Problem: $m_1 = 1$ at $(1, 0)$ and $m_2 = 3$ at $(2, 2)$

Solution:

$$\left(\frac{7}{4}, \frac{3}{2}\right)$$

For the following exercises, compute the center of mass $\bar{x}$.

Exercise:

Problem: $\rho = 1$ for $x \in (-1, 3)$

Exercise:

Problem: $\rho = x^2$ for $x \in (0, L)$

Solution:

$$\frac{3L}{4}$$

Exercise:

Problem: $\rho = 1$ for $x \in (0, 1)$ and $\rho = 2$ for $x \in (1, 2)$

Exercise:

Problem: $\rho = \sin x$ for $x \in (0, \pi)$

Solution:

$$\frac{\pi}{2}$$

Exercise:

Problem: $\rho = \cos x$ for $x \in (0, \frac{\pi}{2})$

Exercise:

Problem: $\rho = e^x$ for $x \in (0, 2)$

Solution:

$$\frac{e^2+1}{e^2-1}$$

Exercise:

Problem: $\rho = x^3 + xe^{-x}$ for $x \in (0, 1)$

Exercise:

Problem: $\rho = x \sin x$ for $x \in (0, \pi)$

Solution:

$$\frac{\pi^2-4}{\pi}$$

Exercise:

Problem: $\rho = \sqrt{x}$ for $x \in (1, 4)$

Exercise:

Problem: $\rho = \ln x$ for $x \in (1, e)$

Solution:

$$\frac{1}{4}(1 + e^2)$$

For the following exercises, compute the center of mass $(\bar{x}, \bar{y})$. Use symmetry to help locate the center of mass whenever possible.

Exercise:

Problem: $\rho = 7$ in the square $0 \leq x \leq 1, 0 \leq y \leq 1$

Exercise:

Problem: $\rho = 3$ in the triangle with vertices $(0, 0)$, $(a, 0)$, and $(0, b)$

Solution:

$$\left(\frac{a}{3}, \frac{b}{3}\right)$$

Exercise:

Problem:

$\rho = 2$ for the region bounded by $y = \cos(x)$, $y = -\cos(x)$, $x = -\frac{\pi}{2}$, and $x = \frac{\pi}{2}$

For the following exercises, use a calculator to draw the region, then compute the center of mass $(\bar{x}, \bar{y})$. Use symmetry to help locate the center of mass whenever possible.

Exercise:

Problem: [T] The region bounded by $y = \cos(2x)$, $x = -\frac{\pi}{4}$, and $x = \frac{\pi}{4}$

Solution:

$$\left(0, \frac{\pi}{8}\right)$$

Exercise:

Problem: [T] The region between $y = 2x^2$, $y = 0$, $x = 0$, and $x = 1$

Exercise:

Problem: [T] The region between $y = \frac{5}{4}x^2$ and $y = 5$

Solution:

$$(0, 3)$$

Exercise:

Problem: [T] Region between $y = \sqrt{x}$, $y = \ln(x)$, $x = 1$, and $x = 4$

Exercise:

Problem: [T] The region bounded by $y = 0$, $\frac{x^2}{4} + \frac{y^2}{9} = 1$

Solution:

$$\left(0, \frac{4}{\pi}\right)$$

Exercise:

Problem: [T] The region bounded by $y = 0$, $x = 0$, and $\frac{x^2}{4} + \frac{y^2}{9} = 1$

Exercise:

Problem: [T] The region bounded by $y = x^2$ and $y = x^4$ in the first quadrant

Solution:

$$\left(\frac{5}{8}, \frac{1}{3}\right)$$

For the following exercises, use the theorem of Pappus to determine the volume of the shape.

Exercise:

Problem: Rotating $y = mx$ around the x -axis between $x = 0$ and $x = 1$

Exercise:

Problem: Rotating $y = mx$ around the y -axis between $x = 0$ and $x = 1$

Solution:

$$\frac{m\pi}{3}$$

Exercise:

Problem:

A general cone created by rotating a triangle with vertices $(0, 0)$, $(a, 0)$, and $(0, b)$ around the y -axis. Does your answer agree with the volume of a cone?

Exercise:

Problem:

A general cylinder created by rotating a rectangle with vertices $(0, 0)$, $(a, 0)$, $(0, b)$, and (a, b) around the y -axis. Does your answer agree with the volume of a cylinder?

Solution:

$$\pi a^2 b$$

Exercise:**Problem:**

A sphere created by rotating a semicircle with radius a around the y -axis. Does your answer agree with the volume of a sphere?

For the following exercises, use a calculator to draw the region enclosed by the curve. Find the area M and the centroid $(\bar{x}, \bar{y})$ for the given shapes. Use symmetry to help locate the center of mass whenever possible.

Exercise:

Problem: [T] Quarter-circle: $y = \sqrt{1 - x^2}$, $y = 0$, and $x = 0$

Solution:

$$\left(\frac{4}{3\pi}, \frac{4}{3\pi} \right)$$

Exercise:

Problem: [T] Triangle: $y = x$, $y = 2 - x$, and $y = 0$

Exercise:

Problem: [T] Lens: $y = x^2$ and $y = x$

Solution:

$$\left(\frac{1}{2}, \frac{2}{5} \right)$$

Exercise:

Problem: [T] Ring: $y^2 + x^2 = 1$ and $y^2 + x^2 = 4$

Exercise:

Problem: [T] Half-ring: $y^2 + x^2 = 1$, $y^2 + x^2 = 4$, and $y = 0$

Solution:

$$\left(0, \frac{28}{9\pi}\right)$$

Exercise:

Problem:

Find the generalized center of mass in the sliver between $y = x^a$ and $y = x^b$ with $a > b$. Then, use the Pappus theorem to find the volume of the solid generated when revolving around the y -axis.

Exercise:

Problem:

Find the generalized center of mass between $y = a^2 - x^2$, $x = 0$, and $y = 0$. Then, use the Pappus theorem to find the volume of the solid generated when revolving around the y -axis.

Solution:

$$\text{Center of mass: } \left(\frac{a}{6}, \frac{4a^2}{5}\right), \text{ volume: } \frac{2\pi a^4}{9}$$

Exercise:

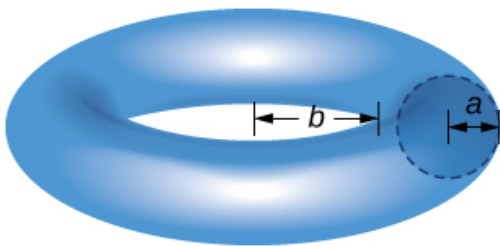
Problem:

Find the generalized center of mass between $y = b \sin(ax)$, $x = 0$, and $x = \frac{\pi}{a}$. Then, use the Pappus theorem to find the volume of the solid generated when revolving around the y -axis.

Exercise:

Problem:

Use the theorem of Pappus to find the volume of a torus (pictured here). Assume that a disk of radius a is positioned with the left end of the circle at $x = b$, $b > 0$, and is rotated around the y -axis.



Solution:

Volume: $2\pi^2 a^2 (b + a)$

Exercise:

Problem:

Find the center of mass $(\bar{x}, \bar{y})$ for a thin wire along the semicircle $y = \sqrt{1 - x^2}$ with unit mass. (*Hint: Use the theorem of Pappus.*)

Glossary

center of mass

the point at which the total mass of the system could be concentrated without changing the moment

centroid

the centroid of a region is the geometric center of the region; laminas are often represented by regions in the plane; if the lamina has a constant density, the center of mass of the lamina depends only on the shape of the corresponding planar region; in this case, the center of mass of the lamina corresponds to the centroid of the representative region

lamina

a thin sheet of material; laminas are thin enough that, for mathematical purposes, they can be treated as if they are two-dimensional

moment

if n masses are arranged on a number line, the moment of the system with respect to the origin is given by $M = \sum_{i=1}^n m_i x_i$; if, instead, we consider a region in the plane, bounded above by a function $f(x)$ over an interval $[a, b]$, then the moments of the region with respect to the x - and y -axes are given by $M_x = \rho \int_a^b \frac{[f(x)]^2}{2} dx$ and

$$M_y = \rho \int_a^b x f(x) dx, \text{ respectively}$$

symmetry principle

the symmetry principle states that if a region R is symmetric about a line l , then the centroid of R lies on l

theorem of Pappus for volume

this theorem states that the volume of a solid of revolution formed by revolving a region around an external axis is equal to the area of the region multiplied by the distance traveled by the centroid of the region

Integrals, Exponential Functions, and Logarithms

- Write the definition of the natural logarithm as an integral.
- Recognize the derivative of the natural logarithm.
- Integrate functions involving the natural logarithmic function.
- Define the number e through an integral.
- Recognize the derivative and integral of the exponential function.
- Prove properties of logarithms and exponential functions using integrals.
- Express general logarithmic and exponential functions in terms of natural logarithms and exponentials.

We already examined exponential functions and logarithms in earlier chapters. However, we glossed over some key details in the previous discussions. For example, we did not study how to treat exponential functions with exponents that are irrational. The definition of the number e is another area where the previous development was somewhat incomplete. We now have the tools to deal with these concepts in a more mathematically rigorous way, and we do so in this section.

For purposes of this section, assume we have not yet defined the natural logarithm, the number e , or any of the integration and differentiation formulas associated with these functions. By the end of the section, we will have studied these concepts in a mathematically rigorous way (and we will see they are consistent with the concepts we learned earlier).

We begin the section by defining the natural logarithm in terms of an integral. This definition forms the foundation for the section. From this definition, we derive differentiation formulas, define the number e , and expand these concepts to logarithms and exponential functions of any base.

The Natural Logarithm as an Integral

Recall the power rule for integrals:

Equation:

$$\int x^n dx = \frac{x^{n+1}}{n+1} + C, \quad n \neq -1.$$

Clearly, this does not work when $n = -1$, as it would force us to divide by zero. So, what do we do with $\int \frac{1}{x} dx$? Recall from the Fundamental Theorem of Calculus that $\int_1^x \frac{1}{t} dt$ is an antiderivative of $1/x$. Therefore, we can make the following definition.

Note:

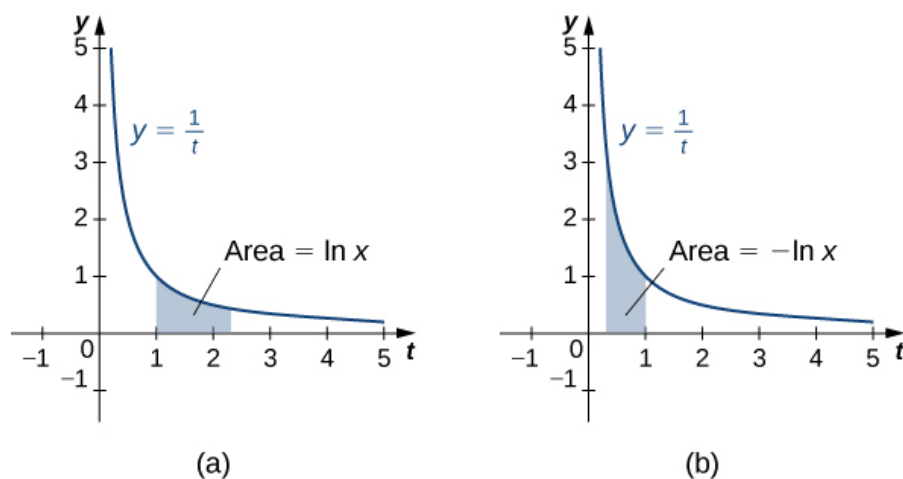
Definition

For $x > 0$, define the natural logarithm function by

Equation:

$$\ln x = \int_1^x \frac{1}{t} dt.$$

For $x > 1$, this is just the area under the curve $y = 1/t$ from 1 to x . For $x < 1$, we have $\int_1^x \frac{1}{t} dt = -\int_x^1 \frac{1}{t} dt$, so in this case it is the negative of the area under the curve from x to 1 (see the following figure).



(a) When $x > 1$, the natural logarithm is the area under the curve $y = 1/t$ from 1 to x . (b) When $x < 1$, the natural logarithm is the negative of the area under the curve from x to 1.

Notice that $\ln 1 = 0$. Furthermore, the function $y = 1/t > 0$ for $x > 0$. Therefore, by the properties of integrals, it is clear that $\ln x$ is increasing for $x > 0$.

Properties of the Natural Logarithm

Because of the way we defined the natural logarithm, the following differentiation formula falls out immediately as a result of the Fundamental Theorem of Calculus.

Note:

Derivative of the Natural Logarithm

For $x > 0$, the derivative of the natural logarithm is given by

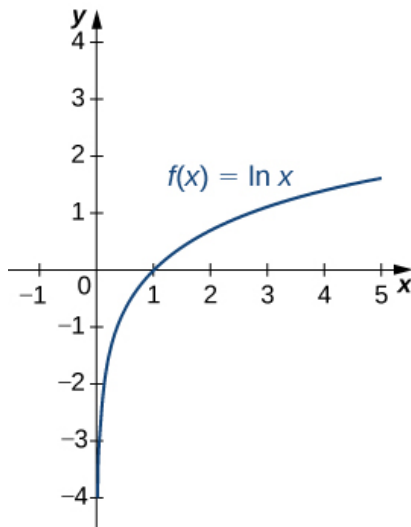
Equation:

$$\frac{d}{dx} \ln x = \frac{1}{x}.$$

Note:**Corollary to the Derivative of the Natural Logarithm**

The function $\ln x$ is differentiable; therefore, it is continuous.

A graph of $\ln x$ is shown in [\[link\]](#). Notice that it is continuous throughout its domain of $(0, \infty)$.



The graph of $f(x) = \ln x$
shows that it is a
continuous function.

Example:**Exercise:****Problem:****Calculating Derivatives of Natural Logarithms**

Calculate the following derivatives:

- a. $\frac{d}{dx} \ln(5x^3 - 2)$
- b. $\frac{d}{dx} (\ln(3x))^2$

Solution:

We need to apply the chain rule in both cases.

$$\begin{aligned} \text{a. } \frac{d}{dx} \ln(5x^3 - 2) &= \frac{15x^2}{5x^3 - 2} \\ \text{b. } \frac{d}{dx} (\ln(3x))^2 &= \frac{2(\ln(3x)) \cdot 3}{3x} = \frac{2(\ln(3x))}{x} \end{aligned}$$

Note:

Exercise:

Problem: Calculate the following derivatives:

$$\begin{aligned} \text{a. } \frac{d}{dx} \ln(2x^2 + x) \\ \text{b. } \frac{d}{dx} (\ln(x^3))^2 \end{aligned}$$

Solution:

$$\begin{aligned} \text{a. } \frac{d}{dx} \ln(2x^2 + x) &= \frac{4x+1}{2x^2+x} \\ \text{b. } \frac{d}{dx} (\ln(x^3))^2 &= \frac{6 \ln(x^3)}{x} \end{aligned}$$

Hint

Apply the differentiation formula just provided and use the chain rule as necessary.

Note that if we use the absolute value function and create a new function $\ln |x|$, we can extend the domain of the natural logarithm to include $x < 0$. Then $(d/(dx))\ln |x| = 1/x$. This gives rise to the familiar integration formula.

Note:

Integral of $(1/u) du$

The natural logarithm is the antiderivative of the function $f(u) = 1/u$:

Equation:

$$\int \frac{1}{u} du = \ln |u| + C.$$

Example:

Exercise:

Problem:

Calculating Integrals Involving Natural Logarithms

Calculate the integral $\int \frac{x}{x^2 + 4} dx$.

Solution:

Using u -substitution, let $u = x^2 + 4$. Then $du = 2x dx$ and we have

Equation:

$$\int \frac{x}{x^2 + 4} dx = \frac{1}{2} \int \frac{1}{u} du = \frac{1}{2} \ln |u| + C = \frac{1}{2} \ln |x^2 + 4| + C = \frac{1}{2} \ln (x^2 + 4) + C.$$

Note:

Exercise:

Problem: Calculate the integral $\int \frac{x^2}{x^3 + 6} dx$.

Solution:

$$\int \frac{x^2}{x^3 + 6} dx = \frac{1}{3} \ln |x^3 + 6| + C$$

Hint

Apply the integration formula provided earlier and use u -substitution as necessary.

Although we have called our function a “logarithm,” we have not actually proved that any of the properties of logarithms hold for this function. We do so here.

Note:

Properties of the Natural Logarithm

If $a, b > 0$ and r is a rational number, then

- i. $\ln 1 = 0$
- ii. $\ln(ab) = \ln a + \ln b$
- iii. $\ln\left(\frac{a}{b}\right) = \ln a - \ln b$
- iv. $\ln(a^r) = r \ln a$

Proof

i. By definition, $\ln 1 = \int_1^1 \frac{1}{t} dt = 0$.

ii. We have

Equation:

$$\ln(ab) = \int_1^{ab} \frac{1}{t} dt = \int_1^a \frac{1}{t} dt + \int_a^{ab} \frac{1}{t} dt.$$

Use u -substitution on the last integral in this expression. Let $u = t/a$. Then $du = (1/a)dt$.

Furthermore, when $t = a$, $u = 1$, and when $t = ab$, $u = b$. So we get

Equation:

$$\ln(ab) = \int_1^a \frac{1}{t} dt + \int_a^{ab} \frac{1}{t} dt = \int_1^a \frac{1}{t} dt + \int_a^{ab} \frac{a}{t} \cdot \frac{1}{a} dt = \int_1^a \frac{1}{t} dt + \int_1^b \frac{1}{u} du = \ln a + \ln b.$$

iv. Note that

Equation:

$$\frac{d}{dx} \ln(x^r) = \frac{rx^{r-1}}{x^r} = \frac{r}{x}.$$

Furthermore,

Equation:

$$\frac{d}{dx} (r \ln x) = \frac{r}{x}.$$

Since the derivatives of these two functions are the same, by the Fundamental Theorem of Calculus, they must differ by a constant. So we have

Equation:

$$\ln(x^r) = r \ln x + C$$

for some constant C . Taking $x = 1$, we get

Equation:

$$\begin{aligned} \ln(1^r) &= r \ln(1) + C \\ 0 &= r(0) + C \\ C &= 0. \end{aligned}$$

Thus $\ln(x^r) = r \ln x$ and the proof is complete. Note that we can extend this property to irrational values of r later in this section.

Part iii. follows from parts ii. and iv. and the proof is left to you.

□

Example:

Exercise:

Problem:

Using Properties of Logarithms

Use properties of logarithms to simplify the following expression into a single logarithm:

Equation:

$$\ln 9 - 2 \ln 3 + \ln \left(\frac{1}{3} \right).$$

Solution:

We have

Equation:

$$\ln 9 - 2 \ln 3 + \ln \left(\frac{1}{3} \right) = \ln(3^2) - 2 \ln 3 + \ln(3^{-1}) = 2 \ln 3 - 2 \ln 3 - \ln 3 = -\ln 3.$$

Note:

Exercise:

Problem:

Use properties of logarithms to simplify the following expression into a single logarithm:

Equation:

$$\ln 8 - \ln 2 - \ln \left(\frac{1}{4} \right).$$

Solution:

$$4 \ln 2$$

Hint

Apply the properties of logarithms.

Defining the Number e

Now that we have the natural logarithm defined, we can use that function to define the number e .

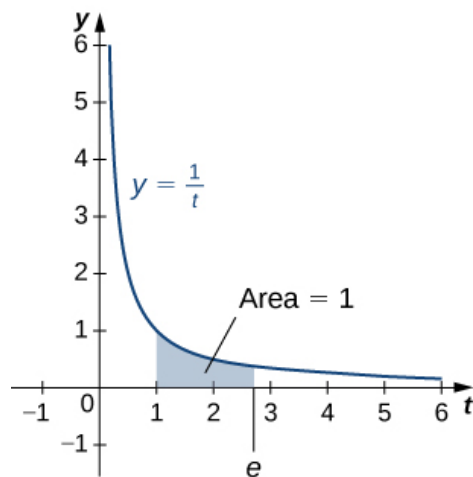
Note:**Definition**

The number e is defined to be the real number such that

Equation:

$$\ln e = 1.$$

To put it another way, the area under the curve $y = 1/t$ between $t = 1$ and $t = e$ is 1 ([\[link\]](#)). The proof that such a number exists and is unique is left to you. (*Hint: Use the Intermediate Value Theorem to prove existence and the fact that $\ln x$ is increasing to prove uniqueness.*)



The area under the curve from 1 to e is equal to one.

The number e can be shown to be irrational, although we won't do so here (see the Student Project in [Taylor and Maclaurin Series](#)). Its approximate value is given by

Equation:

$$e \approx 2.71828182846.$$

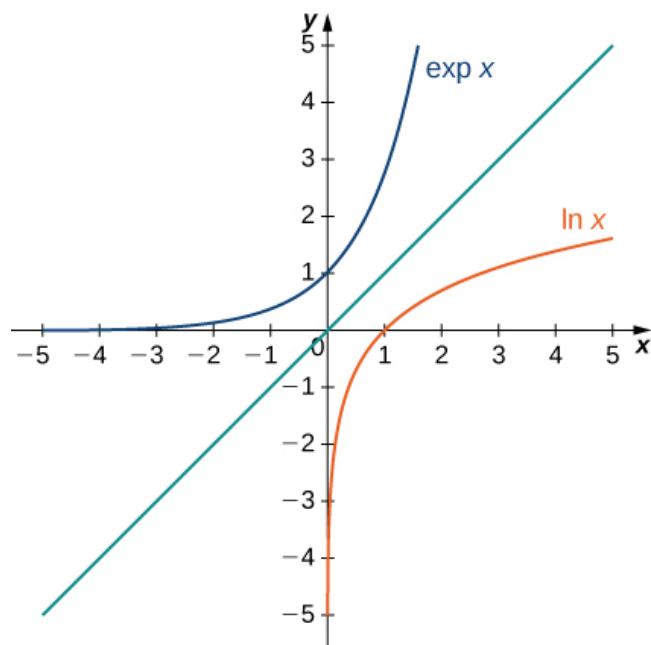
The Exponential Function

We now turn our attention to the function e^x . Note that the natural logarithm is one-to-one and therefore has an inverse function. For now, we denote this inverse function by $\exp x$. Then,

Equation:

$$\exp(\ln x) = x \text{ for } x > 0 \text{ and } \ln(\exp x) = x \text{ for all } x.$$

The following figure shows the graphs of $\exp x$ and $\ln x$.



The graphs of $\ln x$ and $\exp x$.

We hypothesize that $\exp x = e^x$. For rational values of x , this is easy to show. If x is rational, then we have $\ln(e^x) = x \ln e = x$. Thus, when x is rational, $e^x = \exp x$. For irrational values of x , we simply define e^x as the inverse function of $\ln x$.

Note:

Definition

For any real number x , define $y = e^x$ to be the number for which

Equation:

$$\ln y = \ln(e^x) = x.$$

Then we have $e^x = \exp(x)$ for all x , and thus

Equation:

$$e^{\ln x} = x \text{ for } x > 0 \text{ and } \ln(e^x) = x$$

for all x .

Properties of the Exponential Function

Since the exponential function was defined in terms of an inverse function, and not in terms of a power of e , we must verify that the usual laws of exponents hold for the function e^x .

Note:

Properties of the Exponential Function

If p and q are any real numbers and r is a rational number, then

- i. $e^p e^q = e^{p+q}$
- ii. $\frac{e^p}{e^q} = e^{p-q}$
- iii. $(e^p)^r = e^{pr}$

Proof

Note that if p and q are rational, the properties hold. However, if p or q are irrational, we must apply the inverse function definition of e^x and verify the properties. Only the first property is verified here; the other two are left to you. We have

Equation:

$$\ln(e^p e^q) = \ln(e^p) + \ln(e^q) = p + q = \ln(e^{p+q}).$$

Since $\ln x$ is one-to-one, then

Equation:

$$e^p e^q = e^{p+q}.$$

□

As with part iv. of the logarithm properties, we can extend property iii. to irrational values of r , and we do so by the end of the section.

We also want to verify the differentiation formula for the function $y = e^x$. To do this, we need to use implicit differentiation. Let $y = e^x$. Then

Equation:

$$\begin{aligned}\ln y &= x \\ \frac{d}{dx} \ln y &= \frac{d}{dx} x \\ \frac{1}{y} \frac{dy}{dx} &= 1 \\ \frac{dy}{dx} &= y.\end{aligned}$$

Thus, we see

Equation:

$$\frac{d}{dx} e^x = e^x$$

as desired, which leads immediately to the integration formula

Equation:

$$\int e^x dx = e^x + C.$$

We apply these formulas in the following examples.

Example:

Exercise:

Problem:

Using Properties of Exponential Functions

Evaluate the following derivatives:

- a. $\frac{d}{dt} e^{3t} e^{t^2}$
- b. $\frac{d}{dx} e^{3x^2}$

Solution:

We apply the chain rule as necessary.

- a. $\frac{d}{dt} e^{3t} e^{t^2} = \frac{d}{dt} e^{3t+t^2} = e^{3t+t^2} (3 + 2t)$
- b. $\frac{d}{dx} e^{3x^2} = e^{3x^2} 6x$

Note:

Exercise:

Problem: Evaluate the following derivatives:

a. $\frac{d}{dx} \left(\frac{e^{x^2}}{e^{5x}} \right)$

b. $\frac{d}{dt} (e^{2t})^3$

Solution:

a. $\frac{d}{dx} \left(\frac{e^{x^2}}{e^{5x}} \right) = e^{x^2-5x} (2x - 5)$

b. $\frac{d}{dt} (e^{2t})^3 = 6e^{6t}$

Hint

Use the properties of exponential functions and the chain rule as necessary.

Example:

Exercise:

Problem:

Using Properties of Exponential Functions

Evaluate the following integral: $\int 2xe^{-x^2} dx$.

Solution:

Using u -substitution, let $u = -x^2$. Then $du = -2x dx$, and we have

Equation:

$$\int 2xe^{-x^2} dx = - \int e^u du = -e^u + C = -e^{-x^2} + C.$$

Note:

Exercise:

Problem: Evaluate the following integral: $\int \frac{4}{e^{3x}} dx$.

Solution:

$$\int \frac{4}{e^{3x}} dx = -\frac{4}{3}e^{-3x} + C$$

Hint

Use the properties of exponential functions and u -substitution as necessary.

General Logarithmic and Exponential Functions

We close this section by looking at exponential functions and logarithms with bases other than e . Exponential functions are functions of the form $f(x) = a^x$. Note that unless $a = e$, we still do not have a mathematically rigorous definition of these functions for irrational exponents. Let's rectify that here by defining the function $f(x) = a^x$ in terms of the exponential function e^x . We then examine logarithms with bases other than e as inverse functions of exponential functions.

Note:

Definition

For any $a > 0$, and for any real number x , define $y = a^x$ as follows:

Equation:

$$y = a^x = e^{x \ln a}.$$

Now a^x is defined rigorously for all values of x . This definition also allows us to generalize property iv. of logarithms and property iii. of exponential functions to apply to both rational and irrational values of r . It is straightforward to show that properties of exponents hold for general exponential functions defined in this way.

Let's now apply this definition to calculate a differentiation formula for a^x . We have

Equation:

$$\frac{d}{dx} a^x = \frac{d}{dx} e^{x \ln a} = e^{x \ln a} \ln a = a^x \ln a.$$

The corresponding integration formula follows immediately.

Note:

Derivatives and Integrals Involving General Exponential Functions

Let $a > 0$. Then,

Equation:

$$\frac{d}{dx} a^x = a^x \ln a$$

and

Equation:

$$\int a^x dx = \frac{1}{\ln a} a^x + C.$$

If $a \neq 1$, then the function a^x is one-to-one and has a well-defined inverse. Its inverse is denoted by $\log_a x$. Then,

Equation:

$$y = \log_a x \text{ if and only if } x = a^y.$$

Note that general logarithm functions can be written in terms of the natural logarithm. Let $y = \log_a x$. Then, $x = a^y$. Taking the natural logarithm of both sides of this second equation, we get

Equation:

$$\begin{aligned}\ln x &= \ln(a^y) \\ \ln x &= y \ln a \\ y &= \frac{\ln x}{\ln a} \\ \log_a x &= \frac{\ln x}{\ln a}.\end{aligned}$$

Thus, we see that all logarithmic functions are constant multiples of one another. Next, we use this formula to find a differentiation formula for a logarithm with base a . Again, let $y = \log_a x$. Then,

Equation:

$$\begin{aligned}\frac{dy}{dx} &= \frac{d}{dx}(\log_a x) \\ &= \frac{d}{dx}\left(\frac{\ln x}{\ln a}\right) \\ &= \left(\frac{1}{\ln a}\right) \frac{d}{dx}(\ln x) \\ &= \frac{1}{\ln a} \cdot \frac{1}{x} \\ &= \frac{1}{x \ln a}.\end{aligned}$$

Note:

Derivatives of General Logarithm Functions

Let $a > 0$. Then,

Equation:

$$\frac{d}{dx} \log_a x = \frac{1}{x \ln a}.$$

Example:

Exercise:

Problem:

Calculating Derivatives of General Exponential and Logarithm Functions

Evaluate the following derivatives:

a. $\frac{d}{dt} (4^t \cdot 2^{t^2})$

b. $\frac{d}{dx} \log_8 (7x^2 + 4)$

Solution:

We need to apply the chain rule as necessary.

a. $\frac{d}{dt} (4^t \cdot 2^{t^2}) = \frac{d}{dt} (2^{2t} \cdot 2^{t^2}) = \frac{d}{dt} (2^{2t+t^2}) = 2^{2t+t^2} \ln(2) (2 + 2t)$

b. $\frac{d}{dx} \log_8 (7x^2 + 4) = \frac{1}{(7x^2+4)(\ln 8)} (14x)$

Note:

Exercise:

Problem: Evaluate the following derivatives:

a. $\frac{d}{dt} 4^{t^4}$

b. $\frac{d}{dx} \log_3 (\sqrt{x^2 + 1})$

Solution:

a. $\frac{d}{dt} 4^{t^4} = 4^{t^4} (\ln 4) (4t^3)$

b. $\frac{d}{dx} \log_3 (\sqrt{x^2 + 1}) = \frac{x}{(\ln 3)(x^2+1)}$

Hint

Use the formulas and apply the chain rule as necessary.

Example:

Exercise:

Problem:

Integrating General Exponential Functions

Evaluate the following integral: $\int \frac{3}{2^{3x}} dx$.

Solution:

Use u -substitution and let $u = -3x$. Then $du = -3dx$ and we have

Equation:

$$\int \frac{3}{2^{3x}} dx = \int 3 \cdot 2^{-3x} dx = - \int 2^u du = -\frac{1}{\ln 2} 2^u + C = -\frac{1}{\ln 2} 2^{-3x} + C.$$

Note:

Exercise:

Problem: Evaluate the following integral: $\int x^2 2^{x^3} dx$.

Solution:

$$\int x^2 2^{x^3} dx = \frac{1}{3 \ln 2} 2^{x^3} + C$$

Hint

Use the properties of exponential functions and u -substitution as necessary.

Key Concepts

- The earlier treatment of logarithms and exponential functions did not define the functions precisely and formally. This section develops the concepts in a mathematically rigorous way.
- The cornerstone of the development is the definition of the natural logarithm in terms of an integral.
- The function e^x is then defined as the inverse of the natural logarithm.
- General exponential functions are defined in terms of e^x , and the corresponding inverse functions are general logarithms.
- Familiar properties of logarithms and exponents still hold in this more rigorous context.

Key Equations

- **Natural logarithm function**

- $\ln x = \int_1^x \frac{1}{t} dt$ Z

- **Exponential function** $y = e^x$
- $\ln y = \ln(e^x) = x$

For the following exercises, find the derivative $\frac{dy}{dx}$.

Exercise:

Problem: $y = \ln(2x)$

Solution:

$$\frac{1}{x}$$

Exercise:

Problem: $y = \ln(2x + 1)$

Exercise:

Problem: $y = \frac{1}{\ln x}$

Solution:

$$-\frac{1}{x(\ln x)^2}$$

For the following exercises, find the indefinite integral.

Exercise:

Problem: $\int \frac{dt}{3t}$

Exercise:

Problem: $\int \frac{dx}{1+x}$

Solution:

$$\ln(x + 1) + C$$

For the following exercises, find the derivative dy/dx . (You can use a calculator to plot the function and the derivative to confirm that it is correct.)

Exercise:

Problem: [T] $y = \frac{\ln(x)}{x}$

Exercise:

Problem: [T] $y = x \ln(x)$

Solution:

$$\ln(x) + 1$$

Exercise:

Problem: [T] $y = \log_{10} x$

Exercise:

Problem: [T] $y = \ln(\sin x)$

Solution:

$$\cot(x)$$

Exercise:

Problem: [T] $y = \ln(\ln x)$

Exercise:

Problem: [T] $y = 7 \ln(4x)$

Solution:

$$\frac{7}{x}$$

Exercise:

Problem: [T] $y = \ln\left((4x)^7\right)$

Exercise:

Problem: [T] $y = \ln(\tan x)$

Solution:

$$\csc(x) \sec x$$

Exercise:

Problem: [T] $y = \ln(\tan(3x))$

Exercise:

Problem: [T] $y = \ln(\cos^2 x)$

Solution:

$$-2 \tan x$$

For the following exercises, find the definite or indefinite integral.

Exercise:

Problem: $\int_0^1 \frac{dx}{3+x}$

Exercise:

Problem: $\int_0^1 \frac{dt}{3+2t}$

Solution:

$$\frac{1}{2} \ln\left(\frac{5}{3}\right)$$

Exercise:

Problem: $\int_0^2 \frac{x \, dx}{x^2+1}$

Exercise:

Problem: $\int_0^2 \frac{x^3 \, dx}{x^2+1}$

Solution:

$$2 - \frac{1}{2} \ln(5)$$

Exercise:

Problem: $\int_2^e \frac{dx}{x \ln x}$

Exercise:

Problem: $\int_2^e \frac{dx}{x (\ln x)^2}$

Solution:

$$\frac{1}{\ln(2)} - 1$$

Exercise:

Problem: $\int \frac{\cos x \, dx}{\sin x}$

Exercise:

Problem: $\int_0^{\pi/4} \tan x \, dx$

Solution:

$$\frac{1}{2} \ln(2)$$

Exercise:

Problem: $\int \cot(3x) \, dx$

Exercise:

Problem: $\int \frac{(\ln x)^2 \, dx}{x}$

Solution:

$$\frac{1}{3} (\ln x)^3$$

For the following exercises, compute dy/dx by differentiating $\ln y$.

Exercise:

Problem: $y = \sqrt{x^2 + 1}$

Exercise:

Problem: $y = \sqrt{x^2 + 1} \sqrt{x^2 - 1}$

Solution:

$$\frac{2x^3}{\sqrt{x^2+1}\sqrt{x^2-1}}$$

Exercise:

Problem: $y = e^{\sin x}$

Exercise:

Problem: $y = x^{-1/x}$

Solution:

$$x^{-2-(1/x)} (\ln x - 1)$$

Exercise:

Problem: $y = e^{(ex)}$

Exercise:

Problem: $y = x^e$

Solution:

$$ex^{e-1}$$

Exercise:

Problem: $y = x^{(ex)}$

Exercise:

Problem: $y = \sqrt{x} \sqrt[3]{x} \sqrt[6]{x}$

Solution:

$$1$$

Exercise:

Problem: $y = x^{-1/\ln x}$

Exercise:

Problem: $y = e^{-\ln x}$

Solution:

$$-\frac{1}{x^2}$$

For the following exercises, evaluate by any method.

Exercise:

Problem: $\int_5^{10} \frac{dt}{t} - \int_{5x}^{10x} \frac{dt}{t}$

Exercise:

Problem: $\int_1^{e^\pi} \frac{dx}{x} + \int_{-2}^{-1} \frac{dx}{x}$

Solution:

$$\pi - \ln(2)$$

Exercise:

Problem: $\frac{d}{dx} \int_x^1 \frac{dt}{t}$

Exercise:

Problem: $\frac{d}{dx} \int_x^{x^2} \frac{dt}{t}$

Solution:

$$\frac{1}{x}$$

Exercise:

Problem: $\frac{d}{dx} \ln(\sec x + \tan x)$

For the following exercises, use the function $\ln x$. If you are unable to find intersection points analytically, use a calculator.

Exercise:

Problem: Find the area of the region enclosed by $x = 1$ and $y = 5$ above $y = \ln x$.

Solution:

$$e^5 - 6 \text{ units}^2$$

Exercise:

Problem: [T] Find the arc length of $\ln x$ from $x = 1$ to $x = 2$.

Exercise:

Problem: Find the area between $\ln x$ and the x -axis from $x = 1$ to $x = 2$.

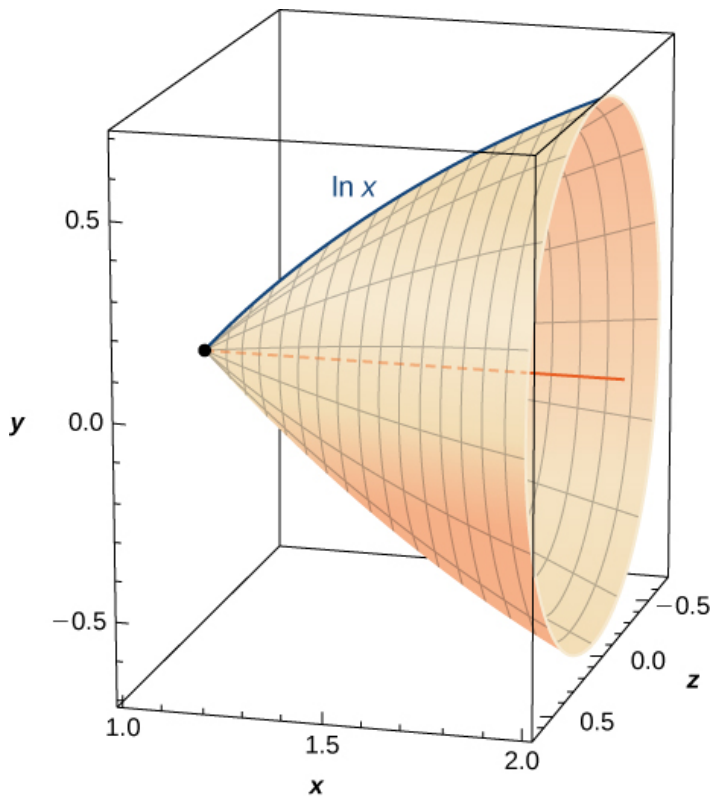
Solution:

$$\ln(4) - 1 \text{ units}^2$$

Exercise:

Problem:

Find the volume of the shape created when rotating this curve from $x = 1$ to $x = 2$ around the x -axis, as pictured here.

**Exercise:****Problem:**

[T] Find the surface area of the shape created when rotating the curve in the previous exercise from $x = 1$ to $x = 2$ around the x -axis.

Solution:

2.8656

If you are unable to find intersection points analytically in the following exercises, use a calculator.

Exercise:**Problem:**

Find the area of the hyperbolic quarter-circle enclosed by $x = 2$ and $y = 2$ above $y = 1/x$.

Exercise:

Problem: [T] Find the arc length of $y = 1/x$ from $x = 1$ to $x = 4$.

Solution:

3.1502

Exercise:

Problem: Find the area under $y = 1/x$ and above the x -axis from $x = 1$ to $x = 4$.

For the following exercises, verify the derivatives and antiderivatives.

Exercise:

Problem: $\frac{d}{dx} \ln \left(x + \sqrt{x^2 + 1} \right) = \frac{1}{\sqrt{1+x^2}}$

Exercise:

Problem: $\frac{d}{dx} \ln \left(\frac{x-a}{x+a} \right) = \frac{2a}{(x^2-a^2)}$

Exercise:

Problem: $\frac{d}{dx} \ln \left(\frac{1+\sqrt{1-x^2}}{x} \right) = -\frac{1}{x\sqrt{1-x^2}}$

Exercise:

Problem: $\frac{d}{dx} \ln \left(x + \sqrt{x^2 - a^2} \right) = \frac{1}{\sqrt{x^2 - a^2}}$

Exercise:

Problem: $\int \frac{dx}{x \ln(x) \ln(\ln x)} = \ln(\ln(\ln x)) + C$

Exponential Growth and Decay

- Use the exponential growth model in applications, including population growth and compound interest.
- Explain the concept of doubling time.
- Use the exponential decay model in applications, including radioactive decay and Newton's law of cooling.
- Explain the concept of half-life.

One of the most prevalent applications of exponential functions involves growth and decay models. Exponential growth and decay show up in a host of natural applications. From population growth and continuously compounded interest to radioactive decay and Newton's law of cooling, exponential functions are ubiquitous in nature. In this section, we examine exponential growth and decay in the context of some of these applications.

Exponential Growth Model

Many systems exhibit exponential growth. These systems follow a model of the form $y = y_0 e^{kt}$, where y_0 represents the initial state of the system and k is a positive constant, called the *growth constant*. Notice that in an exponential growth model, we have

Equation:

$$y' = ky_0 e^{kt} = ky.$$

That is, the rate of growth is proportional to the current function value. This is a key feature of exponential growth. [\[link\]](#) involves derivatives and is called a *differential equation*. We learn more about differential equations in [Introduction to Differential Equations](#).

Note:

Rule: Exponential Growth Model

Systems that exhibit **exponential growth** increase according to the mathematical model

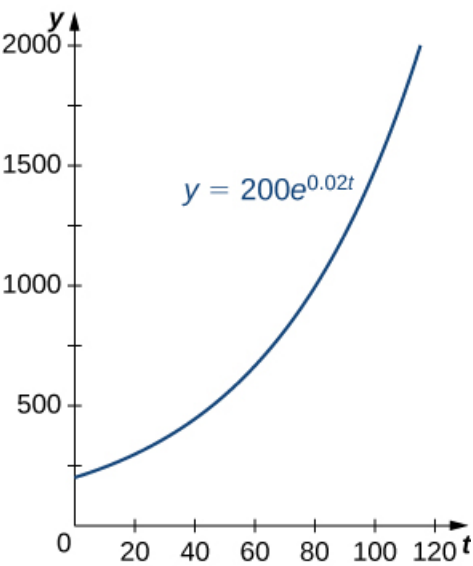
Equation:

$$y = y_0 e^{kt},$$

where y_0 represents the initial state of the system and $k > 0$ is a constant, called the *growth constant*.

Population growth is a common example of exponential growth. Consider a population of bacteria, for instance. It seems plausible that the rate of population growth would be proportional to the size of the population. After all, the more bacteria there are to reproduce, the faster the population grows. [\[link\]](#) and [\[link\]](#) represent the growth of a population of

bacteria with an initial population of 200 bacteria and a growth constant of 0.02. Notice that after only 2 hours (120 minutes), the population is 10 times its original size!



An example of exponential growth for bacteria.

Time (min)	Population Size (no. of bacteria)
10	244
20	298
30	364
40	445
50	544
60	664
70	811

Time (min)	Population Size (no. of bacteria)
80	991
90	1210
100	1478
110	1805
120	2205

Exponential Growth of a Bacterial Population

Note that we are using a continuous function to model what is inherently discrete behavior. At any given time, the real-world population contains a whole number of bacteria, although the model takes on noninteger values. When using exponential growth models, we must always be careful to interpret the function values in the context of the phenomenon we are modeling.

Example:

Exercise:

Problem:

Population Growth

Consider the population of bacteria described earlier. This population grows according to the function $f(t) = 200e^{0.02t}$, where t is measured in minutes. How many bacteria are present in the population after 5 hours (300 minutes)? When does the population reach 100,000 bacteria?

Solution:

We have $f(t) = 200e^{0.02t}$. Then

Equation:

$$f(300) = 200e^{0.02(300)} \approx 80,686.$$

There are 80,686 bacteria in the population after 5 hours.

To find when the population reaches 100,000 bacteria, we solve the equation

Equation:

$$\begin{aligned}
 100,000 &= 200e^{0.02t} \\
 500 &= e^{0.02t} \\
 \ln 500 &= 0.02t \\
 t &= \frac{\ln 500}{0.02} \approx 310.73.
 \end{aligned}$$

The population reaches 100,000 bacteria after 310.73 minutes.

Note:

Exercise:

Problem:

Consider a population of bacteria that grows according to the function $f(t) = 500e^{0.05t}$, where t is measured in minutes. How many bacteria are present in the population after 4 hours? When does the population reach 100 million bacteria?

Solution:

There are 81,377,396 bacteria in the population after 4 hours. The population reaches 100 million bacteria after 244.12 minutes.

Hint

Use the process from the previous example.

Let's now turn our attention to a financial application: compound interest. Interest that is not compounded is called *simple interest*. Simple interest is paid once, at the end of the specified time period (usually 1 year). So, if we put \$1000 in a savings account earning 2% simple interest per year, then at the end of the year we have

Equation:

$$1000(1 + 0.02) = \$1020.$$

Compound interest is paid multiple times per year, depending on the compounding period. Therefore, if the bank compounds the interest every 6 months, it credits half of the year's interest to the account after 6 months. During the second half of the year, the account earns interest not only on the initial \$1000, but also on the interest earned during the first half of the year. Mathematically speaking, at the end of the year, we have

Equation:

$$1000 \left(1 + \frac{0.02}{2} \right)^2 = \$1020.10.$$

Similarly, if the interest is compounded every 4 months, we have

Equation:

$$1000 \left(1 + \frac{0.02}{3} \right)^3 = \$1020.13,$$

and if the interest is compounded daily (365 times per year), we have \$1020.20. If we extend this concept, so that the interest is compounded continuously, after t years we have

Equation:

$$1000 \lim_{n \rightarrow \infty} \left(1 + \frac{0.02}{n} \right)^{nt}.$$

Now let's manipulate this expression so that we have an exponential growth function. Recall that the number e can be expressed as a limit:

Equation:

$$e = \lim_{m \rightarrow \infty} \left(1 + \frac{1}{m} \right)^m.$$

Based on this, we want the expression inside the parentheses to have the form $(1 + 1/m)$. Let $n = 0.02m$. Note that as $n \rightarrow \infty$, $m \rightarrow \infty$ as well. Then we get

Equation:

$$1000 \lim_{n \rightarrow \infty} \left(1 + \frac{0.02}{n} \right)^{nt} = 1000 \lim_{m \rightarrow \infty} \left(1 + \frac{0.02}{0.02m} \right)^{0.02mt} = 1000 \left[\lim_{m \rightarrow \infty} \left(1 + \frac{1}{m} \right)^m \right]^{0.02t}.$$

We recognize the limit inside the brackets as the number e . So, the balance in our bank account after t years is given by $1000e^{0.02t}$. Generalizing this concept, we see that if a bank account with an initial balance of $\$P$ earns interest at a rate of $r\%$, compounded continuously, then the balance of the account after t years is

Equation:

$$\text{Balance} = Pe^{rt}.$$

Example:

Exercise:**Problem:**
Compound Interest

A 25-year-old student is offered an opportunity to invest some money in a retirement account that pays 5% annual interest compounded continuously. How much does the student need to invest today to have \$1 million when she retires at age 65? What if she could earn 6% annual interest compounded continuously instead?

Solution:

We have

Equation:

$$\begin{aligned}1,000,000 &= Pe^{0.05(40)} \\ P &= 135,335.28.\end{aligned}$$

She must invest \$135,335.28 at 5% interest.

If, instead, she is able to earn 6%, then the equation becomes

Equation:

$$\begin{aligned}1,000,000 &= Pe^{0.06(40)} \\ P &= 90,717.95.\end{aligned}$$

In this case, she needs to invest only \$90,717.95. This is roughly two-thirds the amount she needs to invest at 5%. The fact that the interest is compounded continuously greatly magnifies the effect of the 1% increase in interest rate.

Note:**Exercise:****Problem:**

Suppose instead of investing at age 25, the student waits until age 35. How much would she have to invest at 5%? At 6%?

Solution:

At 5% interest, she must invest \$223,130.16. At 6% interest, she must invest \$165,298.89.

Hint

Use the process from the previous example.

If a quantity grows exponentially, the time it takes for the quantity to double remains constant. In other words, it takes the same amount of time for a population of bacteria to grow from 100 to 200 bacteria as it does to grow from 10,000 to 20,000 bacteria. This time is called the doubling time. To calculate the doubling time, we want to know when the quantity reaches twice its original size. So we have

Equation:

$$2y_0 = y_0 e^{kt}$$

$$2 = e^{kt}$$

$$\ln 2 = kt$$

$$t = \frac{\ln 2}{k}.$$

Note:

Definition

If a quantity grows exponentially, the **doubling time** is the amount of time it takes the quantity to double. It is given by

Equation:

$$\text{Doubling time} = \frac{\ln 2}{k}.$$

Example:

Exercise:

Problem:

Using the Doubling Time

Assume a population of fish grows exponentially. A pond is stocked initially with 500 fish. After 6 months, there are 1000 fish in the pond. The owner will allow his friends and neighbors to fish on his pond after the fish population reaches 10,000. When will the owner's friends be allowed to fish?

Solution:

We know it takes the population of fish 6 months to double in size. So, if t represents time in months, by the doubling-time formula, we have $6 = (\ln 2)/k$. Then,

$k = (\ln 2)/6$. Thus, the population is given by $y = 500e^{((\ln 2)/6)t}$. To figure out when the population reaches 10,000 fish, we must solve the following equation:

Equation:

$$\begin{aligned}10,000 &= 500e^{(\ln 2/6)t} \\20 &= e^{(\ln 2/6)t} \\\ln 20 &= \left(\frac{\ln 2}{6}\right)t \\t &= \frac{6(\ln 20)}{\ln 2} \approx 25.93.\end{aligned}$$

The owner's friends have to wait 25.93 months (a little more than 2 years) to fish in the pond.

Note:

Exercise:

Problem:

Suppose it takes 9 months for the fish population in [\[link\]](#) to reach 1000 fish. Under these circumstances, how long do the owner's friends have to wait?

Solution:

38.90 months

Hint

Use the process from the previous example.

Exponential Decay Model

Exponential functions can also be used to model populations that shrink (from disease, for example), or chemical compounds that break down over time. We say that such systems exhibit exponential decay, rather than exponential growth. The model is nearly the same, except there is a negative sign in the exponent. Thus, for some positive constant k , we have $y = y_0e^{-kt}$.

As with exponential growth, there is a differential equation associated with exponential decay. We have

Equation:

$$y' = -ky_0e^{-kt} = -ky.$$

Note:

Rule: Exponential Decay Model

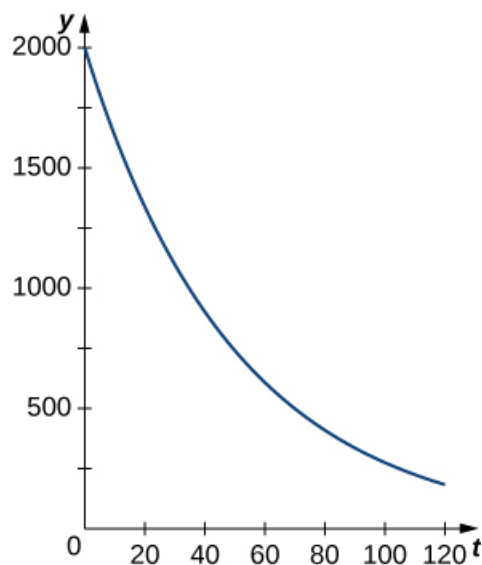
Systems that exhibit **exponential decay** behave according to the model

Equation:

$$y = y_0 e^{-kt},$$

where y_0 represents the initial state of the system and $k > 0$ is a constant, called the *decay constant*.

The following figure shows a graph of a representative exponential decay function.



An example of exponential decay.

Let's look at a physical application of exponential decay. Newton's law of cooling says that an object cools at a rate proportional to the difference between the temperature of the object and the temperature of the surroundings. In other words, if T represents the temperature of the object and T_a represents the ambient temperature in a room, then

Equation:

$$T' = -k(T - T_a).$$

Note that this is not quite the right model for exponential decay. We want the derivative to be proportional to the function, and this expression has the additional T_a term. Fortunately, we can make a change of variables that resolves this issue. Let $y(t) = T(t) - T_a$. Then $y'(t) = T'(t) - 0 = T'(t)$, and our equation becomes

Equation:

$$y' = -ky.$$

From our previous work, we know this relationship between y and its derivative leads to exponential decay. Thus,

Equation:

$$y = y_0 e^{-kt},$$

and we see that

Equation:

$$\begin{aligned} T - T_a &= (T_0 - T_a)e^{-kt} \\ T &= (T_0 - T_a)e^{-kt} + T_a \end{aligned}$$

where T_0 represents the initial temperature. Let's apply this formula in the following example.

Example:

Exercise:

Problem:

Newton's Law of Cooling

According to experienced baristas, the optimal temperature to serve coffee is between 155°F and 175°F . Suppose coffee is poured at a temperature of 200°F , and after 2 minutes in a 70°F room it has cooled to 180°F . When is the coffee first cool enough to serve? When is the coffee too cold to serve? Round answers to the nearest half minute.

Solution:

We have

Equation:

$$\begin{aligned}
 T &= (T_0 - T_a)e^{-kt} + T_a \\
 180 &= (200 - 70)e^{-k(2)} + 70 \\
 110 &= 130e^{-2k} \\
 \frac{11}{13} &= e^{-2k} \\
 \ln \frac{11}{13} &= -2k \\
 \ln 11 - \ln 13 &= -2k \\
 k &= \frac{\ln 13 - \ln 11}{2}.
 \end{aligned}$$

Then, the model is

Equation:

$$T = 130e^{\left(\frac{\ln 11 - \ln 13}{2}\right)t} + 70.$$

The coffee reaches 175 °F when

Equation:

$$\begin{aligned}
 175 &= 130e^{\left(\frac{\ln 11 - \ln 13}{2}\right)t} + 70 \\
 105 &= 130e^{\left(\frac{\ln 11 - \ln 13}{2}\right)t} \\
 \frac{21}{26} &= e^{\left(\frac{\ln 11 - \ln 13}{2}\right)t} \\
 \ln \frac{21}{26} &= \frac{\ln 11 - \ln 13}{2}t \\
 \ln 21 - \ln 26 &= \frac{\ln 11 - \ln 13}{2}t \\
 t &= \frac{2(\ln 21 - \ln 26)}{\ln 11 - \ln 13} \approx 2.56.
 \end{aligned}$$

The coffee can be served about 2.5 minutes after it is poured. The coffee reaches 155 °F at

Equation:

$$\begin{aligned}
 155 &= 130e^{\left(\frac{\ln 11 - \ln 13}{2}\right)t} + 70 \\
 85 &= 130e^{\left(\frac{\ln 11 - \ln 13}{2}\right)t} \\
 \frac{17}{26} &= e^{\left(\frac{\ln 11 - \ln 13}{2}\right)t} \\
 \ln 17 - \ln 26 &= \left(\frac{\ln 11 - \ln 13}{2}\right)t \\
 t &= \frac{2(\ln 17 - \ln 26)}{\ln 11 - \ln 13} \approx 5.09.
 \end{aligned}$$

The coffee is too cold to be served about 5 minutes after it is poured.

Note:**Exercise:****Problem:**

Suppose the room is warmer (75°F) and, after 2 minutes, the coffee has cooled only to 185°F . When is the coffee first cool enough to serve? When is the coffee too cold to serve? Round answers to the nearest half minute.

Solution:

The coffee is first cool enough to serve about 3.5 minutes after it is poured. The coffee is too cold to serve about 7 minutes after it is poured.

Hint

Use the process from the previous example.

Just as systems exhibiting exponential growth have a constant doubling time, systems exhibiting exponential decay have a constant half-life. To calculate the half-life, we want to know when the quantity reaches half its original size. Therefore, we have

Equation:

$$\begin{aligned}\frac{y_0}{2} &= y_0 e^{-kt} \\ \frac{1}{2} &= e^{-kt} \\ -\ln 2 &= -kt \\ t &= \frac{\ln 2}{k}.\end{aligned}$$

Note: This is the same expression we came up with for doubling time.

Note:**Definition**

If a quantity decays exponentially, the **half-life** is the amount of time it takes the quantity to be reduced by half. It is given by

Equation:

$$\text{Half-life} = \frac{\ln 2}{k}.$$

Example:

Exercise:**Problem:****Radiocarbon Dating**

One of the most common applications of an exponential decay model is carbon dating. Carbon-14 decays (emits a radioactive particle) at a regular and consistent exponential rate. Therefore, if we know how much carbon was originally present in an object and how much carbon remains, we can determine the age of the object. The half-life of carbon-14 is approximately 5730 years—meaning, after that many years, half the material has converted from the original carbon-14 to the new nonradioactive nitrogen-14. If we have 100 g carbon-14 today, how much is left in 50 years? If an artifact that originally contained 100 g of carbon now contains 10 g of carbon, how old is it? Round the answer to the nearest hundred years.

Solution:

We have

Equation:

$$\begin{aligned} 5730 &= \frac{\ln 2}{k} \\ k &= \frac{\ln 2}{5730}. \end{aligned}$$

So, the model says

Equation:

$$y = 100e^{-(\ln 2/5730)t}.$$

In 50 years, we have

Equation:

$$\begin{aligned} y &= 100e^{-(\ln 2/5730)(50)} \\ &\approx 99.40. \end{aligned}$$

Therefore, in 50 years, 99.40 g of carbon-14 remains.

To determine the age of the artifact, we must solve

Equation:

$$\begin{aligned} 10 &= 100e^{-(\ln 2/5730)t} \\ \frac{1}{10} &= e^{-(\ln 2/5730)t} \\ t &\approx 19035. \end{aligned}$$

The artifact is about 19,000 years old.

Note:

Exercise:

Problem:

If we have 100 g of carbon-14, how much is left after t years? If an artifact that originally contained 100 g of carbon now contains 20g of carbon, how old is it? Round the answer to the nearest hundred years.

Solution:

A total of 94.13 g of carbon remains. The artifact is approximately 13,300 years old.

Hint

Use the process from the previous example.

Key Concepts

- Exponential growth and exponential decay are two of the most common applications of exponential functions.
- Systems that exhibit exponential growth follow a model of the form $y = y_0 e^{kt}$.
- In exponential growth, the rate of growth is proportional to the quantity present. In other words, $y' = ky$.
- Systems that exhibit exponential growth have a constant doubling time, which is given by $(\ln 2)/k$.
- Systems that exhibit exponential decay follow a model of the form $y = y_0 e^{-kt}$.
- Systems that exhibit exponential decay have a constant half-life, which is given by $(\ln 2)/k$.

True or False? If true, prove it. If false, find the true answer.

Exercise:

Problem: The doubling time for $y = e^{ct}$ is $(\ln(2))/(\ln(c))$.

Exercise:

Problem:

If you invest \$500, an annual rate of interest of 3% yields more money in the first year than a 2.5% continuous rate of interest.

Solution:

True

Exercise:

Problem:

If you leave a 100°C pot of tea at room temperature (25°C) and an identical pot in the refrigerator (5°C), with $k = 0.02$, the tea in the refrigerator reaches a drinkable temperature (70°C) more than 5 minutes before the tea at room temperature.

Exercise:

Problem:

If given a half-life of t years, the constant k for $y = e^{kt}$ is calculated by $k = \ln(1/2)/t$.

Solution:

False; $k = \frac{\ln(2)}{t}$

For the following exercises, use $y = y_0e^{kt}$.

Exercise:

Problem:

If a culture of bacteria doubles in 3 hours, how many hours does it take to multiply by 10?

Exercise:

Problem:

If bacteria increase by a factor of 10 in 10 hours, how many hours does it take to increase by 100?

Solution:

20 hours

Exercise:

Problem:

How old is a skull that contains one-fifth as much radiocarbon as a modern skull? Note that the half-life of radiocarbon is 5730 years.

Exercise:

Problem:

If a relic contains 90% as much radiocarbon as new material, can it have come from the time of Christ (approximately 2000 years ago)? Note that the half-life of radiocarbon is 5730 years.

Solution:

No. The relic is approximately 871 years old.

Exercise:**Problem:**

The population of Cairo grew from 5 million to 10 million in 20 years. Use an exponential model to find when the population was 8 million.

Exercise:**Problem:**

The populations of New York and Los Angeles are growing at 1% and 1.4% a year, respectively. Starting from 8 million (New York) and 6 million (Los Angeles), when are the populations equal?

Solution:

71.92 years

Exercise:**Problem:**

Suppose the value of \$1 in Japanese yen decreases at 2% per year. Starting from \$1 = ¥250, when will \$1 = ¥1?

Exercise:**Problem:**

The effect of advertising decays exponentially. If 40% of the population remembers a new product after 3 days, how long will 20% remember it?

Solution:

5 days 6 hours 27 minutes

Exercise:

Problem: If $y = 1000$ at $t = 3$ and $y = 3000$ at $t = 4$, what was y_0 at $t = 0$?

Exercise:

Problem: If $y = 100$ at $t = 4$ and $y = 10$ at $t = 8$, when does $y = 1$?

Solution:

12

Exercise:

Problem:

If a bank offers annual interest of 7.5% or continuous interest of 7.25%, which has a better annual yield?

Exercise:

Problem: What continuous interest rate has the same yield as an annual rate of 9%?

Solution:

8.618%

Exercise:

Problem:

If you deposit \$5000 at 8% annual interest, how many years can you withdraw \$500 (starting after the first year) without running out of money?

Exercise:

Problem:

You are trying to save \$50,000 in 20 years for college tuition for your child. If interest is a continuous 10%, how much do you need to invest initially?

Solution:

\$6766.76

Exercise:

Problem:

You are cooling a turkey that was taken out of the oven with an internal temperature of 165°F . After 10 minutes of resting the turkey in a 70°F apartment, the temperature has reached 155°F . What is the temperature of the turkey 20 minutes after taking it out of the oven?

Exercise:

Problem:

You are trying to thaw some vegetables that are at a temperature of 1°F . To thaw vegetables safely, you must put them in the refrigerator, which has an ambient temperature of 44°F . You check on your vegetables 2 hours after putting them in the refrigerator to find that they are now 12°F . Plot the resulting temperature curve and use it to determine when the vegetables reach 33°F .

Solution:

9 hours 13 minutes

Exercise:

Problem:

You are an archaeologist and are given a bone that is claimed to be from a Tyrannosaurus Rex. You know these dinosaurs lived during the Cretaceous Era (146 million years to 65 million years ago), and you find by radiocarbon dating that there is 0.000001% the amount of radiocarbon. Is this bone from the Cretaceous?

Exercise:

Problem:

The spent fuel of a nuclear reactor contains plutonium-239, which has a half-life of 24,000 years. If 1 barrel containing 10 kg of plutonium-239 is sealed, how many years must pass until only 10g of plutonium-239 is left?

Solution:

239,179 years

For the next set of exercises, use the following table, which features the world population by decade.

Years since 1950	Population (millions)
0	2,556
10	3,039
20	3,706
30	4,453
40	5,279
50	6,083
60	6,849

Source: <http://www.factmonster.com/ipka/A0762181.html>.

Exercise:

Problem:

[T] The best-fit exponential curve to the data of the form $P(t) = ae^{bt}$ is given by $P(t) = 2686e^{0.01604t}$. Use a graphing calculator to graph the data and the exponential curve together.

Exercise:

Problem:

[T] Find and graph the derivative y' of your equation. Where is it increasing and what is the meaning of this increase?

Solution:

$P'(t) = 43e^{0.01604t}$. The population is always increasing.

Exercise:

Problem:

[T] Find and graph the second derivative of your equation. Where is it increasing and what is the meaning of this increase?

Exercise:

Problem:

[T] Find the predicted date when the population reaches 10 billion. Using your previous answers about the first and second derivatives, explain why exponential growth is unsuccessful in predicting the future.

Solution:

The population reaches 10 billion people in 2027.

For the next set of exercises, use the following table, which shows the population of San Francisco during the 19th century.

Years since 1850	Population (thousands)
0	21.00

Years since 1850	Population (thousands)
10	56.80
20	149.5
30	234.0

Source: <http://www.sfgenealogy.com/sf/history/hgpop.htm>.

Exercise:

Problem:

[T] The best-fit exponential curve to the data of the form $P(t) = ae^{bt}$ is given by $P(t) = 35.26e^{0.06407t}$. Use a graphing calculator to graph the data and the exponential curve together.

Exercise:

Problem:

[T] Find and graph the derivative y' of your equation. Where is it increasing? What is the meaning of this increase? Is there a value where the increase is maximal?

Solution:

$P'(t) = 2.259e^{0.06407t}$. The population is always increasing.

Exercise:

Problem:

[T] Find and graph the second derivative of your equation. Where is it increasing? What is the meaning of this increase?

Glossary

doubling time

if a quantity grows exponentially, the doubling time is the amount of time it takes the quantity to double, and is given by $(\ln 2)/k$

exponential decay

systems that exhibit exponential decay follow a model of the form $y = y_0e^{-kt}$

exponential growth

systems that exhibit exponential growth follow a model of the form $y = y_0e^{kt}$

half-life

if a quantity decays exponentially, the half-life is the amount of time it takes the quantity to be reduced by half. It is given by $(\ln 2)/k$

Integration by Parts

- Recognize when to use integration by parts.
- Use the integration-by-parts formula to solve integration problems.
- Use the integration-by-parts formula for definite integrals.

By now we have a fairly thorough procedure for how to evaluate many basic integrals. However, although we can integrate $\int x \sin(x^2) dx$ by using the substitution, $u = x^2$, something as simple looking as $\int x \sin x dx$ defies us.

Many students want to know whether there is a product rule for integration. There isn't, but there is a technique based on the product rule for differentiation that allows us to exchange one integral for another. We call this technique **integration by parts**.

The Integration-by-Parts Formula

If, $h(x) = f(x)g(x)$, then by using the product rule, we obtain $h'(x) = f'(x)g(x) + g'(x)f(x)$. Although at first it may seem counterproductive, let's now integrate both sides of this equation:

$$\int h'(x) dx = \int (g(x)f'(x) + f(x)g'(x)) dx.$$

This gives us

Equation:

$$h(x) = f(x)g(x) = \int g(x)f'(x) dx + \int f(x)g'(x) dx.$$

Now we solve for $\int f(x)g'(x) dx$:

Equation:

$$\int f(x)g'(x) dx = f(x)g(x) - \int g(x)f'(x) dx.$$

By making the substitutions $u = f(x)$ and $v = g(x)$, which in turn make $du = f'(x)dx$ and $dv = g'(x)dx$, we have the more compact form

Equation:

$$\int u dv = uv - \int v du.$$

Note:

Integration by Parts

Let $u = f(x)$ and $v = g(x)$ be functions with continuous derivatives. Then, the integration-by-parts formula for the integral involving these two functions is:

Equation:

$$\int u dv = uv - \int v du.$$

The advantage of using the integration-by-parts formula is that we can use it to exchange one integral for another, possibly easier, integral. The following example illustrates its use.

Example:

Exercise:

Problem:

Using Integration by Parts

Use integration by parts with $u = x$ and $dv = \sin x \, dx$ to evaluate $\int x \sin x \, dx$.

Solution:

By choosing $u = x$, we have $du = 1dx$. Since $dv = \sin x \, dx$, we get $v = \int \sin x \, dx = -\cos x$. It is handy to keep track of these values as follows:

Equation:

$$\begin{array}{ll} u &= x & dv &= \sin x \, dx \\ du &= 1dx & v &= \int \sin x \, dx = -\cos x. \end{array}$$

Applying the integration-by-parts formula results in

Equation:

$$\begin{aligned} \int x \sin x \, dx &= (x)(-\cos x) - \int (-\cos x)(1dx) && \text{Substitute.} \\ &= -x \cos x + \int \cos x \, dx && \text{Simplify.} \\ &= -x \cos x + \sin x + C. && \text{Use } \int \cos x \, dx = \sin x + C. \end{aligned}$$

Analysis

At this point, there are probably a few items that need clarification. First of all, you may be curious about what would have happened if we had chosen $u = \sin x$ and $dv = x$. If we had done so, then we would have $du = \cos x \, dx$ and $v = \frac{1}{2}x^2$. Thus, after applying integration by parts, we have

$\int x \sin x \, dx = \frac{1}{2}x^2 \sin x - \int \frac{1}{2}x^2 \cos x \, dx$. Unfortunately, with the new integral, we are in no better position than before. It is important to keep in mind that when we apply integration by parts, we may need to try several choices for u and dv before finding a choice that works.

Second, you may wonder why, when we find $v = \int \sin x \, dx = -\cos x$, we do not use $v = -\cos x + K$. To see that it makes no difference, we can rework the problem using $v = -\cos x + K$:

Equation:

$$\begin{aligned}
\int x \sin x \, dx &= (x)(-\cos x + K) - \int (-\cos x + K)(1 \, dx) \\
&= -x \cos x + Kx + \int \cos x \, dx - \int K \, dx \\
&= -x \cos x + Kx + \sin x - Kx + C \\
&= -x \cos x + \sin x + C.
\end{aligned}$$

As you can see, it makes no difference in the final solution.

Last, we can check to make sure that our antiderivative is correct by differentiating $-x \cos x + \sin x + C$:

Equation:

$$\begin{aligned}
\frac{d}{dx}(-x \cos x + \sin x + C) &= (-1)\cos x + (-x)(-\sin x) + \cos x \\
&= x \sin x.
\end{aligned}$$

Therefore, the antiderivative checks out.

Note:

Watch this [video](#) and visit this [website](#) for examples of integration by parts.

Note:

Exercise:

Problem: Evaluate $\int x e^{2x} \, dx$ using the integration-by-parts formula with $u = x$ and $dv = e^{2x} \, dx$.

Solution:

$$\int x e^{2x} \, dx = \frac{1}{2} x e^{2x} - \frac{1}{4} e^{2x} + C$$

Hint

Find du and v , and use the previous example as a guide.

The natural question to ask at this point is: How do we know how to choose u and dv ? Sometimes it is a matter of trial and error; however, the acronym LIATE can often help to take some of the guesswork out of our choices. This acronym stands for **L**ogarithmic Functions, **I**nverse Trigonometric Functions, **A**lgebraic Functions, **T**rigonometric Functions, and **E**xponential Functions. This mnemonic serves as an aid in determining an appropriate choice for u .

The type of function in the integral that appears first in the list should be our first choice of u . For example, if an integral contains a logarithmic function and an algebraic function, we should choose u to be the logarithmic function, because L comes before A in LIATE. The integral in [\[link\]](#) has a trigonometric function ($\sin x$) and an algebraic function (x). Because A comes before T in LIATE, we chose u to be the algebraic function. When we have chosen u , dv is selected to be the remaining part of the function to be integrated, together with dx .

Why does this mnemonic work? Remember that whatever we pick to be dv must be something we can integrate. Since we do not have integration formulas that allow us to integrate simple logarithmic functions and inverse trigonometric functions, it makes sense that they should not be chosen as values for dv . Consequently, they should be at the head of the list as choices for u . Thus, we put LI at the beginning of the mnemonic. (We could just as easily have started with IL, since these two types of functions won't appear together in an integration-by-parts problem.) The exponential and trigonometric functions are at the end of our list because they are fairly easy to integrate and make good choices for dv . Thus, we have TE at the end of our mnemonic. (We could just as easily have used ET at the end, since when these types of functions appear together it usually doesn't really matter which one is u and which one is dv .) Algebraic functions are generally easy both to integrate and to differentiate, and they come in the middle of the mnemonic.

Example:

Exercise:

Problem:

Using Integration by Parts

Evaluate $\int \frac{\ln x}{x^3} dx$.

Solution:

Begin by rewriting the integral:

Equation:

$$\int \frac{\ln x}{x^3} dx = \int x^{-3} \ln x dx.$$

Since this integral contains the algebraic function x^{-3} and the logarithmic function $\ln x$, choose $u = \ln x$, since L comes before A in LIATE. After we have chosen $u = \ln x$, we must choose $dv = x^{-3} dx$.

Next, since $u = \ln x$, we have $du = \frac{1}{x} dx$. Also, $v = \int x^{-3} dx = -\frac{1}{2} x^{-2}$. Summarizing,

Equation:

$$\begin{aligned} u &= \ln x & dv &= x^{-3} dx \\ du &= \frac{1}{x} dx & v &= \int x^{-3} dx = -\frac{1}{2} x^{-2}. \end{aligned}$$

Substituting into the integration-by-parts formula ([link](#)) gives

Equation:

$$\begin{aligned} \int \frac{\ln x}{x^3} dx &= \int x^{-3} \ln x dx = (\ln x) \left(-\frac{1}{2} x^{-2}\right) - \int \left(-\frac{1}{2} x^{-2}\right) \left(\frac{1}{x} dx\right) \\ &= -\frac{1}{2} x^{-2} \ln x + \int \frac{1}{2} x^{-3} dx \\ &= -\frac{1}{2} x^{-2} \ln x - \frac{1}{4} x^{-2} + C \\ &= -\frac{1}{2x^2} \ln x - \frac{1}{4x^2} + C. \end{aligned}$$

Simplify.

Integrate.

Rewrite with positive integers.

Note:

Exercise:

Problem: Evaluate $\int x \ln x \, dx$.

Solution:

$$\frac{1}{2}x^2 \ln x - \frac{1}{4}x^2 + C$$

Hint

Use $u = \ln x$ and $dv = x \, dx$.

In some cases, as in the next two examples, it may be necessary to apply integration by parts more than once.

Example:

Exercise:

Problem:
Applying Integration by Parts More Than Once

Evaluate $\int x^2 e^{3x} \, dx$.

Solution:

Using LIATE, choose $u = x^2$ and $dv = e^{3x} \, dx$. Thus, $du = 2x \, dx$ and $v = \int e^{3x} \, dx = \left(\frac{1}{3}\right)e^{3x}$.

Therefore,

Equation:

$$\begin{aligned} u &= x^2 & dv &= e^{3x} \, dx \\ du &= 2x \, dx & v &= \int e^{3x} \, dx = \frac{1}{3}e^{3x}. \end{aligned}$$

Substituting into [\[link\]](#) produces

Equation:

$$\int x^2 e^{3x} \, dx = \frac{1}{3}x^2 e^{3x} - \int \frac{2}{3}x e^{3x} \, dx.$$

We still cannot integrate $\int \frac{2}{3}x e^{3x} \, dx$ directly, but the integral now has a lower power on x . We can evaluate this new integral by using integration by parts again. To do this, choose $u = x$ and $dv = \frac{2}{3}e^{3x} \, dx$. Thus, $du = dx$ and $v = \int \left(\frac{2}{3}\right)e^{3x} \, dx = \left(\frac{2}{9}\right)e^{3x}$. Now we have

Equation:

$$\begin{aligned} u &= x & dv &= \frac{2}{3}e^{3x}dx \\ du &= dx & v &= \int \frac{2}{3}e^{3x}dx = \frac{2}{9}e^{3x}. \end{aligned}$$

Substituting back into the previous equation yields

Equation:

$$\int x^2 e^{3x} dx = \frac{1}{3}x^2 e^{3x} - \left(\frac{2}{9}x e^{3x} - \int \frac{2}{9}e^{3x} dx \right).$$

After evaluating the last integral and simplifying, we obtain

Equation:

$$\int x^2 e^{3x} dx = \frac{1}{3}x^2 e^{3x} - \frac{2}{9}x e^{3x} + \frac{2}{27}e^{3x} + C.$$

Example:

Exercise:

Problem:

Applying Integration by Parts When LIATE Doesn't Quite Work

Evaluate $\int t^3 e^{t^2} dt$.

Solution:

If we use a strict interpretation of the mnemonic LIATE to make our choice of u , we end up with $u = t^3$ and $dv = e^{t^2} dt$. Unfortunately, this choice won't work because we are unable to evaluate $\int e^{t^2} dt$. However, since we can evaluate $\int t e^{t^2} dx$, we can try choosing $u = t^2$ and $dv = t e^{t^2} dt$. With these choices we have

Equation:

$$\begin{aligned} u &= t^2 & dv &= t e^{t^2} dt \\ du &= 2t dt & v &= \int t e^{t^2} dt = \frac{1}{2}e^{t^2}. \end{aligned}$$

Thus, we obtain

Equation:

$$\begin{aligned} \int t^3 e^{t^2} dt &= \frac{1}{2}t^2 e^{t^2} - \int \frac{1}{2}e^{t^2} 2t dt \\ &= \frac{1}{2}t^2 e^{t^2} - \frac{1}{2}e^{t^2} + C. \end{aligned}$$

Example:

Exercise:**Problem:****Applying Integration by Parts More Than Once**

Evaluate $\int \sin(\ln x) dx$.

Solution:

This integral appears to have only one function—namely, $\sin(\ln x)$ —however, we can always use the constant function 1 as the other function. In this example, let's choose $u = \sin(\ln x)$ and $dv = 1 dx$. (The decision to use $u = \sin(\ln x)$ is easy. We can't choose $dv = \sin(\ln x) dx$ because if we could integrate it, we wouldn't be using integration by parts in the first place!) Consequently, $du = (1/x)\cos(\ln x) dx$ and

$v = \int 1 dx = x$. After applying integration by parts to the integral and simplifying, we have

Equation:

$$\int \sin(\ln x) dx = x \sin(\ln x) - \int \cos(\ln x) dx.$$

Unfortunately, this process leaves us with a new integral that is very similar to the original. However, let's see what happens when we apply integration by parts again. This time let's choose $u = \cos(\ln x)$ and

$dv = 1 dx$, making $du = -(1/x)\sin(\ln x) dx$ and $v = \int 1 dx = x$. Substituting, we have

Equation:

$$\int \sin(\ln x) dx = x \sin(\ln x) - \left(x \cos(\ln x) - \int -\sin(\ln x) dx \right).$$

After simplifying, we obtain

Equation:

$$\int \sin(\ln x) dx = x \sin(\ln x) - x \cos(\ln x) - \int \sin(\ln x) dx.$$

The last integral is now the same as the original. It may seem that we have simply gone in a circle, but now we can actually evaluate the integral. To see how to do this more clearly, substitute $I = \int \sin(\ln x) dx$.

Thus, the equation becomes

Equation:

$$I = x \sin(\ln x) - x \cos(\ln x) - I.$$

First, add I to both sides of the equation to obtain

Equation:

$$2I = x \sin(\ln x) - x \cos(\ln x).$$

Next, divide by 2:

Equation:

$$I = \frac{1}{2}x \sin(\ln x) - \frac{1}{2}x \cos(\ln x).$$

Substituting $I = \int \sin(\ln x) dx$ again, we have

Equation:

$$\int \sin(\ln x) dx = \frac{1}{2}x \sin(\ln x) - \frac{1}{2}x \cos(\ln x).$$

From this we see that $(1/2)x \sin(\ln x) - (1/2)x \cos(\ln x)$ is an antiderivative of $\sin(\ln x) dx$. For the most general antiderivative, add $+C$:

Equation:

$$\int \sin(\ln x) dx = \frac{1}{2}x \sin(\ln x) - \frac{1}{2}x \cos(\ln x) + C.$$

Analysis

If this method feels a little strange at first, we can check the answer by differentiation:

Equation:

$$\begin{aligned} & \frac{d}{dx} \left(\frac{1}{2}x \sin(\ln x) - \frac{1}{2}x \cos(\ln x) \right) \\ &= \frac{1}{2}(\sin(\ln x)) + \cos(\ln x) \cdot \frac{1}{x} \cdot \frac{1}{2}x - \left(\frac{1}{2}\cos(\ln x) - \sin(\ln x) \cdot \frac{1}{x} \cdot \frac{1}{2}x \right) \\ &= \sin(\ln x). \end{aligned}$$

Note:

Exercise:

Problem: Evaluate $\int x^2 \sin x \, dx$.

Solution:

$$-x^2 \cos x + 2x \sin x + 2 \cos x + C$$

Hint

This is similar to [\[link\]](#).

Integration by Parts for Definite Integrals

Now that we have used integration by parts successfully to evaluate indefinite integrals, we turn our attention to definite integrals. The integration technique is really the same, only we add a step to evaluate the integral at the upper and lower limits of integration.

Note:

Integration by Parts for Definite Integrals

Let $u = f(x)$ and $v = g(x)$ be functions with continuous derivatives on $[a, b]$. Then

Equation:

$$\int_a^b u \, dv = uv \Big|_a^b - \int_a^b v \, du.$$

Example:

Exercise:

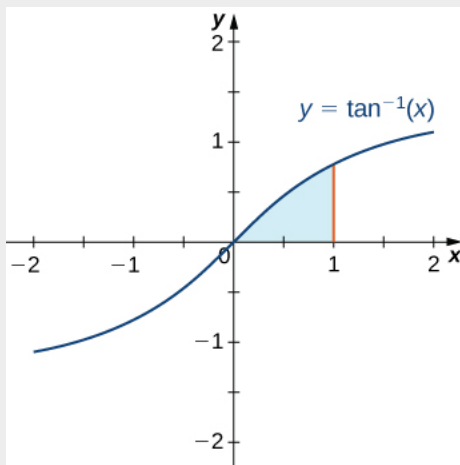
Problem:

Finding the Area of a Region

Find the area of the region bounded above by the graph of $y = \tan^{-1}x$ and below by the x -axis over the interval $[0, 1]$.

Solution:

This region is shown in [\[link\]](#). To find the area, we must evaluate $\int_0^1 \tan^{-1}x \, dx$.



To find the area of the shaded region, we have to use integration by parts.

For this integral, let's choose $u = \tan^{-1}x$ and $dv = dx$, thereby making $du = \frac{1}{x^2+1}dx$ and $v = x$. After applying the integration-by-parts formula ([\[link\]](#)) we obtain

Equation:

$$\text{Area} = x \tan^{-1}x \Big|_0^1 - \int_0^1 \frac{x}{x^2+1} dx.$$

Use u -substitution to obtain

Equation:

$$\int_0^1 \frac{x}{x^2 + 1} dx = \frac{1}{2} \ln|x^2 + 1| \Big|_0^1.$$

Thus,

Equation:

$$\text{Area} = x \tan^{-1} x \Big|_0^1 - \frac{1}{2} \ln|x^2 + 1| \Big|_0^1 = \frac{\pi}{4} - \frac{1}{2} \ln 2.$$

At this point it might not be a bad idea to do a “reality check” on the reasonableness of our solution. Since $\frac{\pi}{4} - \frac{1}{2} \ln 2 \approx 0.4388$, and from [\[link\]](#) we expect our area to be slightly less than 0.5, this solution appears to be reasonable.

Example:

Exercise:

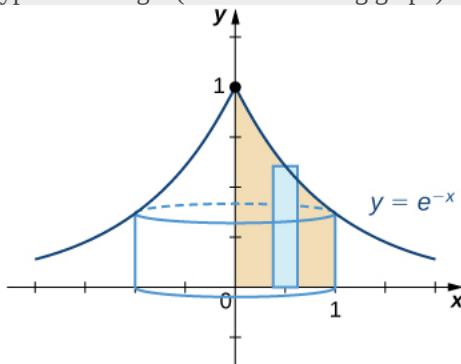
Problem:

Finding a Volume of Revolution

Find the volume of the solid obtained by revolving the region bounded by the graph of $f(x) = e^{-x}$, the x -axis, the y -axis, and the line $x = 1$ about the y -axis.

Solution:

The best option to solving this problem is to use the shell method. Begin by sketching the region to be revolved, along with a typical rectangle (see the following graph).



We can use the shell method to find a volume of revolution.

To find the volume using shells, we must evaluate $2\pi \int_0^1 x e^{-x} dx$. To do this, let $u = x$ and $dv = e^{-x}$.

These choices lead to $du = dx$ and $v = \int e^{-x} = -e^{-x}$. Substituting into [\[link\]](#), we obtain

Equation:

$$\text{Volume} = 2\pi \int_0^1 x e^{-x} dx = 2\pi \left(-x e^{-x} \Big|_0^1 + \int_0^1 e^{-x} dx \right) \quad \text{Use integration by parts.}$$

$$= -2\pi x e^{-x} \Big|_0^1 - 2\pi e^{-x} \Big|_0^1$$

$$= 2\pi - \frac{4\pi}{e}.$$

$$\text{Evaluate } \int_0^1 e^{-x} dx = -e^{-x} \Big|_0^1.$$

Evaluate and simplify.

Analysis

Again, it is a good idea to check the reasonableness of our solution. We observe that the solid has a volume slightly less than that of a cylinder of radius 1 and height of $1/e$ added to the volume of a cone of base radius 1 and height of $1 - \frac{1}{e}$. Consequently, the solid should have a volume a bit less than

Equation:

$$\pi(1)^2 \frac{1}{e} + \left(\frac{\pi}{3}\right)(1)^2 \left(1 - \frac{1}{e}\right) = \frac{2\pi}{3e} + \frac{\pi}{3} \approx 1.8177.$$

Since $2\pi - \frac{4\pi}{e} \approx 1.6603$, we see that our calculated volume is reasonable.

Note:

Exercise:

Problem: Evaluate $\int_0^{\pi/2} x \cos x \, dx$.

Solution:

$$\frac{\pi}{2} - 1$$

Hint

Use [\[link\]](#) with $u = x$ and $dv = \cos x \, dx$.

Key Concepts

- The integration-by-parts formula allows the exchange of one integral for another, possibly easier, integral.
- Integration by parts applies to both definite and indefinite integrals.

Key Equations

- **Integration by parts formula**

$$\int u \, dv = uv - \int v \, du$$

- **Integration by parts for definite integrals**

$$\int_a^b u \, dv = uv \Big|_a^b - \int_a^b v \, du$$

In using the technique of integration by parts, you must carefully choose which expression is u . For each of the following problems, use the guidelines in this section to choose u . Do **not** evaluate the integrals.

Exercise:

Problem: $\int x^3 e^{2x} dx$

Solution:

$$u = x^3$$

Exercise:

Problem: $\int x^3 \ln(x) dx$

Exercise:

Problem: $\int y^3 \cos y dy$

Solution:

$$u = y^3$$

Exercise:

Problem: $\int x^2 \arctan x dx$

Exercise:

Problem: $\int e^{3x} \sin(2x) dx$

Solution:

$$u = \sin(2x)$$

Find the integral by using the simplest method. Not all problems require integration by parts.

Exercise:

Problem: $\int v \sin v dv$

Exercise:

Problem: $\int \ln x dx$ (Hint: $\int \ln x dx$ is equivalent to $\int 1 \cdot \ln(x) dx$.)

Solution:

$$-x + x \ln x + C$$

Exercise:

Problem: $\int x \cos x \, dx$

Exercise:

Problem: $\int \tan^{-1} x \, dx$

Solution:

$$x \tan^{-1} x - \frac{1}{2} \ln(1 + x^2) + C$$

Exercise:

Problem: $\int x^2 e^x \, dx$

Exercise:

Problem: $\int x \sin(2x) \, dx$

Solution:

$$-\frac{1}{2} x \cos(2x) + \frac{1}{4} \sin(2x) + C$$

Exercise:

Problem: $\int x e^{4x} \, dx$

Exercise:

Problem: $\int x e^{-x} \, dx$

Solution:

$$e^{-x}(-1 - x) + C$$

Exercise:

Problem: $\int x \cos 3x \, dx$

Exercise:

Problem: $\int x^2 \cos x \, dx$

Solution:

$$2x \cos x + (-2 + x^2) \sin x + C$$

Exercise:

Problem: $\int x \ln x \, dx$

Exercise:

Problem: $\int \ln(2x + 1) dx$

Solution:

$$\frac{1}{2}(1 + 2x)(-1 + \ln(1 + 2x)) + C$$

Exercise:

Problem: $\int x^2 e^{4x} dx$

Exercise:

Problem: $\int e^x \sin x \, dx$

Solution:

$$\frac{1}{2}e^x(-\cos x + \sin x) + C$$

Exercise:

Problem: $\int e^x \cos x \, dx$

Exercise:

Problem: $\int x e^{-x^2} dx$

Solution:

$$-\frac{e^{-x^2}}{2} + C$$

Exercise:

Problem: $\int x^2 e^{-x} dx$

Exercise:

Problem: $\int \sin(\ln(2x)) dx$

Solution:

$$-\frac{1}{2}x \cos[\ln(2x)] + \frac{1}{2}x \sin[\ln(2x)] + C$$

Exercise:

Problem: $\int \cos(\ln x) dx$

Exercise:

Problem: $\int (\ln x)^2 dx$

Solution:

$$2x - 2x \ln x + x(\ln x)^2 + C$$

Exercise:

Problem: $\int \ln(x^2) dx$

Exercise:

Problem: $\int x^2 \ln x \, dx$

Solution:

$$\left(-\frac{x^3}{9} + \frac{1}{3}x^3 \ln x\right) + C$$

Exercise:

Problem: $\int \sin^{-1} x \, dx$

Exercise:

Problem: $\int \cos^{-1}(2x) dx$

Solution:

$$-\frac{1}{2}\sqrt{1-4x^2} + x \cos^{-1}(2x) + C$$

Exercise:

Problem: $\int x \arctan x \, dx$

Exercise:

Problem: $\int x^2 \sin x \, dx$

Solution:

$$-(-2 + x^2)\cos x + 2x \sin x + C$$

Exercise:

Problem: $\int x^3 \cos x \, dx$

Exercise:

Problem: $\int x^3 \sin x \, dx$

Solution:

$$-x(-6 + x^2)\cos x + 3(-2 + x^2)\sin x + C$$

Exercise:

Problem: $\int x^3 e^x \, dx$

Exercise:

Problem: $\int x \sec^{-1} x \, dx$

Solution:

$$\frac{1}{2}x \left(-\sqrt{1 - \frac{1}{x^2}} + x \cdot \sec^{-1} x \right) + C$$

Exercise:

Problem: $\int x \sec^2 x \, dx$

Exercise:

Problem: $\int x \cosh x \, dx$

Solution:

$$-\cosh x + x \sinh x + C$$

Compute the definite integrals. Use a graphing utility to confirm your answers.

Exercise:

Problem: $\int_{1/e}^1 \ln x \, dx$

Exercise:

Problem: $\int_0^1 x e^{-2x} \, dx$ (Express the answer in exact form.)

Solution:

$$\frac{1}{4} - \frac{3}{4e^2}$$

Exercise:

Problem: $\int_0^1 e^{\sqrt{x}} dx$ (let $u = \sqrt{x}$)

Exercise:

Problem: $\int_1^e \ln(x^2) dx$

Solution:

$$2$$

Exercise:

Problem: $\int_0^\pi x \cos x \, dx$

Exercise:

Problem: $\int_{-\pi}^\pi x \sin x \, dx$ (Express the answer in exact form.)

Solution:

$$2\pi$$

Exercise:

Problem: $\int_0^3 \ln(x^2 + 1) dx$ (Express the answer in exact form.)

Exercise:

Problem: $\int_0^{\pi/2} x^2 \sin x \, dx$ (Express the answer in exact form.)

Solution:

$$-2 + \pi$$

Exercise:

Problem: $\int_0^1 x 5^x dx$ (Express the answer using five significant digits.)

Exercise:

Problem: Evaluate $\int \cos x \ln(\sin x) dx$

Solution:

$$-\sin(x) + \ln[\sin(x)]\sin x + C$$

Derive the following formulas using the technique of integration by parts. Assume that n is a positive integer. These formulas are called *reduction formulas* because the exponent in the x term has been reduced by one in each case. The second integral is simpler than the original integral.

Exercise:

Problem: $\int x^n e^x dx = x^n e^x - n \int x^{n-1} e^x dx$

Exercise:

Problem: $\int x^n \cos x dx = x^n \sin x - n \int x^{n-1} \sin x dx$

Solution:

Answers vary

Exercise:

Problem: $\int x^n \sin x dx = \text{munder}$

Exercise:

Problem: Integrate $\int 2x\sqrt{2x-3} dx$ using two methods:

- Using parts, letting $dv = \sqrt{2x-3} dx$
- Substitution, letting $u = 2x-3$

Solution:

a. $\frac{2}{5}(1+x)(-3+2x)^{3/2} + C$ b. $\frac{2}{5}(1+x)(-3+2x)^{3/2} + C$

State whether you would use integration by parts to evaluate the integral. If so, identify u and dv . If not, describe the technique used to perform the integration without actually doing the problem.

Exercise:

Problem: $\int x \ln x dx$

Exercise:

Problem: $\int \frac{\ln^2 x}{x} dx$

Solution:

Do not use integration by parts. Choose u to be $\ln x$, and the integral is of the form $\int u^2 du$.

Exercise:

Problem: $\int x e^x dx$

Exercise:

Problem: $\int x e^{x^2-3} dx$

Solution:

Do not use integration by parts. Let $u = x^2 - 3$, and the integral can be put into the form $\int e^u du$.

Exercise:

Problem: $\int x^2 \sin x dx$

Exercise:

Problem: $\int x^2 \sin(3x^3 + 2) dx$

Solution:

Do not use integration by parts. Choose u to be $u = 3x^3 + 2$ and the integral can be put into the form $\int \sin(u) du$.

Sketch the region bounded above by the curve, the x-axis, and $x = 1$, and find the area of the region. Provide the exact form or round answers to the number of places indicated.

Exercise:

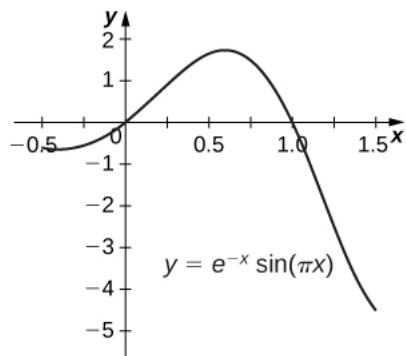
Problem: $y = 2x e^{-x}$ (Approximate answer to four decimal places.)

Exercise:

Problem: $y = e^{-x} \sin(\pi x)$ (Approximate answer to five decimal places.)

Solution:

The area under graph is 0.39535.



Find the volume generated by rotating the region bounded by the given curves about the specified line. Express the answers in exact form or approximate to the number of decimal places indicated.

Exercise:

Problem: $y = \sin x$, $y = 0$, $x = 2\pi$, $x = 3\pi$ about the y-axis (Express the answer in exact form.)

Exercise:

Problem: $y = e^{-x}$, $y = 0$, $x = -1$, $x = 0$; about $x = 1$ (Express the answer in exact form.)

Solution:

$$2\pi e$$

Exercise:

Problem:

A particle moving along a straight line has a velocity of $v(t) = t^2 e^{-t}$ after t sec. How far does it travel in the first 2 sec? (Assume the units are in feet and express the answer in exact form.)

Exercise:

Problem:

Find the area under the graph of $y = \sec^3 x$ from $x = 0$ to $x = 1$. (Round the answer to two significant digits.)

Solution:

$$2.05$$

Exercise:

Problem:

Find the area between $y = (x - 2)e^x$ and the x-axis from $x = 2$ to $x = 5$. (Express the answer in exact form.)

Exercise:

Problem: Find the area of the region enclosed by the curve $y = x \cos x$ and the x-axis for

$$\frac{11\pi}{2} \leq x \leq \frac{13\pi}{2}. \text{ (Express the answer in exact form.)}$$

Solution:

$$12\pi$$

Exercise:

Problem:

Find the volume of the solid generated by revolving the region bounded by the curve $y = \ln x$, the x-axis, and the vertical line $x = e^2$ about the x-axis. (Express the answer in exact form.)

Exercise:

Problem:

Find the volume of the solid generated by revolving the region bounded by the curve $y = 4 \cos x$ and the x -axis, $\frac{\pi}{2} \leq x \leq \frac{3\pi}{2}$, about the x -axis. (Express the answer in exact form.)

Solution:

$$8\pi^2$$

Exercise:**Problem:**

Find the volume of the solid generated by revolving the region in the first quadrant bounded by $y = e^x$ and the x -axis, from $x = 0$ to $x = \ln(7)$, about the y -axis. (Express the answer in exact form.)

Glossary

integration by parts

a technique of integration that allows the exchange of one integral for another using the formula

$$\int u \, dv = uv - \int v \, du$$

Trigonometric Integrals

- Solve integration problems involving products and powers of $\sin x$ and $\cos x$.
- Solve integration problems involving products and powers of $\tan x$ and $\sec x$.
- Use reduction formulas to solve trigonometric integrals.

In this section we look at how to integrate a variety of products of trigonometric functions. These integrals are called **trigonometric integrals**. They are an important part of the integration technique called *trigonometric substitution*, which is featured in [Trigonometric Substitution](#). This technique allows us to convert algebraic expressions that we may not be able to integrate into expressions involving trigonometric functions, which we may be able to integrate using the techniques described in this section. In addition, these types of integrals appear frequently when we study polar, cylindrical, and spherical coordinate systems later. Let's begin our study with products of $\sin x$ and $\cos x$.

Integrating Products and Powers of $\sin x$ and $\cos x$

A key idea behind the strategy used to integrate combinations of products and powers of $\sin x$ and $\cos x$ involves rewriting these expressions as sums and differences of integrals of the form $\int \sin^j x \cos x \, dx$ or $\int \cos^j x \sin x \, dx$. After rewriting these integrals, we evaluate them using u -substitution. Before describing the general process in detail, let's take a look at the following examples.

Example:

Exercise:

Problem:

Integrating $\int \cos^j x \sin x \, dx$

Evaluate $\int \cos^3 x \sin x \, dx$.

Solution:

Use u -substitution and let $u = \cos x$. In this case, $du = -\sin x \, dx$. Thus,

Equation:

$$\begin{aligned}\int \cos^3 x \sin x \, dx &= -\int u^3 \, du \\ &= -\frac{1}{4}u^4 + C \\ &= -\frac{1}{4}\cos^4 x + C.\end{aligned}$$

Note:

Exercise:

Problem: Evaluate $\int \sin^4 x \cos x \, dx$.

Solution:

$$\frac{1}{5} \sin^5 x + C$$

Hint

Let $u = \sin x$.

Example:

Exercise:

Problem:

A Preliminary Example: Integrating $\int \cos^j x \sin^k x \, dx$ Where k is Odd

Evaluate $\int \cos^2 x \sin^3 x \, dx$.

Solution:

To convert this integral to integrals of the form $\int \cos^j x \sin x \, dx$, rewrite $\sin^3 x = \sin^2 x \sin x$ and make the substitution $\sin^2 x = 1 - \cos^2 x$. Thus,

Equation:

$$\begin{aligned} \int \cos^2 x \sin^3 x \, dx &= \int \cos^2 x (1 - \cos^2 x) \sin x \, dx \quad \text{Let } u = \cos x; \text{ then } du = -\sin x \, dx. \\ &= - \int u^2 (1 - u^2) du \\ &= - \int (u^2 - u^4) du \\ &= -\frac{1}{3} u^3 + \frac{1}{5} u^5 + C \\ &= -\frac{1}{3} \cos^3 x + \frac{1}{5} \cos^5 x + C. \end{aligned}$$

Note:

Exercise:

Problem: Evaluate $\int \cos^3 x \sin^2 x \, dx$.

Solution:

$$\frac{1}{3} \sin^3 x - \frac{1}{5} \sin^5 x + C$$

Hint

Write $\cos^3 x = \cos^2 x \cos x = (1 - \sin^2 x) \cos x$ and let $u = \sin x$.

In the next example, we see the strategy that must be applied when there are only even powers of $\sin x$ and $\cos x$. For integrals of this type, the identities

Equation:

$$\sin^2 x = \frac{1}{2} - \frac{1}{2} \cos(2x) = \frac{1 - \cos(2x)}{2}$$

and

Equation:

$$\cos^2 x = \frac{1}{2} + \frac{1}{2} \cos(2x) = \frac{1 + \cos(2x)}{2}$$

are invaluable. These identities are sometimes known as *power-reducing identities* and they may be derived from the double-angle identity $\cos(2x) = \cos^2 x - \sin^2 x$ and the Pythagorean identity $\cos^2 x + \sin^2 x = 1$.

Example:

Exercise:

Problem:

Integrating an Even Power of $\sin x$

Evaluate $\int \sin^2 x \, dx$.

Solution:

To evaluate this integral, let's use the trigonometric identity $\sin^2 x = \frac{1}{2} - \frac{1}{2} \cos(2x)$. Thus,

Equation:

$$\begin{aligned} \int \sin^2 x \, dx &= \int \left(\frac{1}{2} - \frac{1}{2} \cos(2x) \right) dx \\ &= \frac{1}{2} x - \frac{1}{4} \sin(2x) + C. \end{aligned}$$

Note:

Exercise:

Problem: Evaluate $\int \cos^2 x \, dx$.

Solution:

$$\frac{1}{2} x + \frac{1}{4} \sin(2x) + C$$

Hint

Equation:

$$\cos^2 x = \frac{1}{2} + \frac{1}{2}\cos(2x)$$

The general process for integrating products of powers of $\sin x$ and $\cos x$ is summarized in the following set of guidelines.

Note:

Problem-Solving Strategy: Integrating Products and Powers of $\sin x$ and $\cos x$

To integrate $\int \cos^j x \sin^k x \, dx$ use the following strategies:

1. If k is odd, rewrite $\sin^k x = \sin^{k-1} x \sin x$ and use the identity $\sin^2 x = 1 - \cos^2 x$ to rewrite $\sin^{k-1} x$ in terms of $\cos x$. Integrate using the substitution $u = \cos x$. This substitution makes $du = -\sin x \, dx$.
2. If j is odd, rewrite $\cos^j x = \cos^{j-1} x \cos x$ and use the identity $\cos^2 x = 1 - \sin^2 x$ to rewrite $\cos^{j-1} x$ in terms of $\sin x$. Integrate using the substitution $u = \sin x$. This substitution makes $du = \cos x \, dx$. (Note: If both j and k are odd, either strategy 1 or strategy 2 may be used.)
3. If both j and k are even, use $\sin^2 x = (1/2) - (1/2)\cos(2x)$ and $\cos^2 x = (1/2) + (1/2)\cos(2x)$. After applying these formulas, simplify and reapply strategies 1 through 3 as appropriate.

Example:

Exercise:

Problem:

Integrating $\int \cos^j x \sin^k x \, dx$ **where k is Odd**

Evaluate $\int \cos^8 x \sin^5 x \, dx$.

Solution:

Since the power on $\sin x$ is odd, use strategy 1. Thus,

Equation:

$\int \cos^8 x \sin^5 x \, dx = \int \cos^8 x \sin^4 x \sin x \, dx$	Break off $\sin x$.
$= \int \cos^8 x (\sin^2 x)^2 \sin x \, dx$	Rewrite $\sin^4 x = (\sin^2 x)^2$.
$= \int \cos^8 x (1 - \cos^2 x)^2 \sin x \, dx$	Substitute $\sin^2 x = 1 - \cos^2 x$.
$= \int u^8 (1 - u^2)^2 (-du)$	Let $u = \cos x$ and $du = -\sin x \, dx$.
$= \int (-u^8 + 2u^{10} - u^{12}) \, du$	Expand.
$= -\frac{1}{9}u^9 + \frac{2}{11}u^{11} - \frac{1}{13}u^{13} + C$	Evaluate the integral.
$= -\frac{1}{9}\cos^9 x + \frac{2}{11}\cos^{11} x - \frac{1}{13}\cos^{13} x + C.$	Substitute $u = \cos x$.

Example:

Exercise:

Problem:

Integrating $\int \cos^j x \sin^k x \, dx$ **where k and j are Even**

Evaluate $\int \sin^4 x \, dx$.

Solution:

Since the power on $\sin x$ is even ($k = 4$) and the power on $\cos x$ is even ($j = 0$), we must use strategy 3. Thus,

Equation:

$\int \sin^4 x \, dx = \int (\sin^2 x)^2 \, dx$	Rewrite $\sin^4 x = (\sin^2 x)^2$.
$= \int \left(\frac{1}{2} - \frac{1}{2}\cos(2x)\right)^2 \, dx$	Substitute $\sin^2 x = \frac{1}{2} - \frac{1}{2}\cos(2x)$.
$= \int \left(\frac{1}{4} - \frac{1}{2}\cos(2x) + \frac{1}{4}\cos^2(2x)\right) \, dx$	Expand $\left(\frac{1}{2} - \frac{1}{2}\cos(2x)\right)^2$.
$= \int \left(\frac{1}{4} - \frac{1}{2}\cos(2x) + \frac{1}{4}\left(\frac{1}{2} + \frac{1}{2}\cos(4x)\right)\right) \, dx.$	

Since $\cos^2(2x)$ has an even power, substitute $\cos^2(2x) = \frac{1}{2} + \frac{1}{2}\cos(4x)$:

Equation:

$= \int \left(\frac{3}{8} - \frac{1}{2}\cos(2x) + \frac{1}{8}\cos(4x)\right) \, dx$	Simplify.
$= \frac{3}{8}x - \frac{1}{4}\sin(2x) + \frac{1}{32}\sin(4x) + C$	Evaluate the integral.

Note:

Exercise:

Problem: Evaluate $\int \cos^3 x \, dx$.

Solution:

$$\sin x - \frac{1}{3} \sin^3 x + C$$

Hint

Use strategy 2. Write $\cos^3 x = \cos^2 x \cos x$ and substitute $\cos^2 x = 1 - \sin^2 x$.

Note:

Exercise:

Problem: Evaluate $\int \cos^2(3x) dx$.

Solution:

$$\frac{1}{2} x + \frac{1}{12} \sin(6x) + C$$

Hint

Use strategy 3. Substitute $\cos^2(3x) = \frac{1}{2} + \frac{1}{2} \cos(6x)$

In some areas of physics, such as quantum mechanics, signal processing, and the computation of Fourier series, it is often necessary to integrate products that include $\sin(ax)$, $\sin(bx)$, $\cos(ax)$, and $\cos(bx)$. These integrals are evaluated by applying trigonometric identities, as outlined in the following rule.

Note:

Rule: Integrating Products of Sines and Cosines of Different Angles

To integrate products involving $\sin(ax)$, $\sin(bx)$, $\cos(ax)$, and $\cos(bx)$, use the substitutions

Equation:

$$\sin(ax)\sin(bx) = \frac{1}{2}\cos((a-b)x) - \frac{1}{2}\cos((a+b)x)$$

Equation:

$$\sin(ax)\cos(bx) = \frac{1}{2}\sin((a-b)x) + \frac{1}{2}\sin((a+b)x)$$

Equation:

$$\cos(ax)\cos(bx) = \frac{1}{2}\cos((a-b)x) + \frac{1}{2}\cos((a+b)x)$$

These formulas may be derived from the sum-of-angle formulas for sine and cosine.

Example:

Exercise:

Problem:

Evaluating $\int \sin(ax)\cos(bx)dx$

Evaluate $\int \sin(5x)\cos(3x)dx$.

Solution:

Apply the identity $\sin(5x)\cos(3x) = \frac{1}{2}\sin(2x) - \frac{1}{2}\sin(8x)$. Thus,

Equation:

$$\begin{aligned}\int \sin(5x)\cos(3x)dx &= \int \frac{1}{2}\sin(2x)dx + \int \frac{1}{2}\sin(8x)dx \\ &= -\frac{1}{4}\cos(2x) - \frac{1}{16}\cos(8x) + C.\end{aligned}$$

Note:

Exercise:

Problem: Evaluate $\int \cos(6x)\cos(5x)dx$.

Solution:

$$\frac{1}{2}\sin x + \frac{1}{22}\sin(11x) + C$$

Hint

Substitute $\cos(6x)\cos(5x) = \frac{1}{2}\cos x + \frac{1}{2}\cos(11x)$.

Integrating Products and Powers of $\tan x$ and $\sec x$

Before discussing the integration of products and powers of $\tan x$ and $\sec x$, it is useful to recall the integrals involving $\tan x$ and $\sec x$ we have already learned:

1. $\int \sec^2 x dx = \tan x + C$
2. $\int \sec x \tan x dx = \sec x + C$

$$3. \int \tan x \, dx = \ln |\sec x| + C$$

$$4. \int \sec x \, dx = \ln |\sec x + \tan x| + C.$$

For most integrals of products and powers of $\tan x$ and $\sec x$, we rewrite the expression we wish to integrate as the sum or difference of integrals of the form $\int \tan^j x \sec^2 x \, dx$ or $\int \sec^j x \tan x \, dx$. As we see in the following example, we can evaluate these new integrals by using u -substitution.

Example:

Exercise:

Problem:

Evaluating $\int \sec^j x \tan x \, dx$

Evaluate $\int \sec^5 x \tan x \, dx$.

Solution:

Start by rewriting $\sec^5 x \tan x$ as $\sec^4 x \sec x \tan x$.

Equation:

$$\begin{aligned} \int \sec^5 x \tan x \, dx &= \int \sec^4 x \sec x \tan x \, dx && \text{Let } u = \sec x; \text{ then, } du = \sec x \tan x \, dx. \\ &= \int u^4 du && \text{Evaluate the integral.} \\ &= \frac{1}{5} u^5 + C && \text{Substitute } \sec x = u. \\ &= \frac{1}{5} \sec^5 x + C \end{aligned}$$

Note:

You can read some interesting information at this [website](#) to learn about a common integral involving the secant.

Note:

Exercise:

Problem: Evaluate $\int \tan^5 x \sec^2 x \, dx$.

Solution:

$$\frac{1}{6} \tan^6 x + C$$

Hint

Let $u = \tan x$ and $du = \sec^2 x$.

We now take a look at the various strategies for integrating products and powers of $\sec x$ and $\tan x$.

Note:

Problem-Solving Strategy: Integrating $\int \tan^k x \sec^j x \, dx$

To integrate $\int \tan^k x \sec^j x \, dx$, use the following strategies:

1. If j is even and $j \geq 2$, rewrite $\sec^j x = \sec^{j-2} x \sec^2 x$ and use $\sec^2 x = \tan^2 x + 1$ to rewrite $\sec^{j-2} x$ in terms of $\tan x$. Let $u = \tan x$ and $du = \sec^2 x \, dx$.
2. If k is odd and $j \geq 1$, rewrite $\tan^k x \sec^j x = \tan^{k-1} x \sec^{j-1} x \sec x \tan x$ and use $\tan^2 x = \sec^2 x - 1$ to rewrite $\tan^{k-1} x$ in terms of $\sec x$. Let $u = \sec x$ and $du = \sec x \tan x \, dx$. (Note: If j is even and k is odd, then either strategy 1 or strategy 2 may be used.)
3. If k is odd where $k \geq 3$ and $j = 0$, rewrite $\tan^k x = \tan^{k-2} x \tan^2 x = \tan^{k-2} x (\sec^2 x - 1) = \tan^{k-2} x \sec^2 x - \tan^{k-2} x$. It may be necessary to repeat this process on the $\tan^{k-2} x$ term.
4. If k is even and j is odd, then use $\tan^2 x = \sec^2 x - 1$ to express $\tan^k x$ in terms of $\sec x$. Use integration by parts to integrate odd powers of $\sec x$.

Example:**Exercise:****Problem:**

Integrating $\int \tan^k x \sec^j x \, dx$ **when j is Even**

Evaluate $\int \tan^6 x \sec^4 x \, dx$.

Solution:

Since the power on $\sec x$ is even, rewrite $\sec^4 x = \sec^2 x \sec^2 x$ and use $\sec^2 x = \tan^2 x + 1$ to rewrite the first $\sec^2 x$ in terms of $\tan x$. Thus,

Equation:

$$\begin{aligned}
 \int \tan^6 x \sec^4 x \, dx &= \int \tan^6 x (\tan^2 x + 1) \sec^2 x \, dx && \text{Let } u = \tan x \text{ and } du = \sec^2 x \, dx. \\
 &= \int u^6 (u^2 + 1) \, du && \text{Expand.} \\
 &= \int (u^8 + u^6) \, du && \text{Evaluate the integral.} \\
 &= \frac{1}{9} u^9 + \frac{1}{7} u^7 + C && \text{Substitute } \tan x = u. \\
 &= \frac{1}{9} \tan^9 x + \frac{1}{7} \tan^7 x + C.
 \end{aligned}$$

Example:

Exercise:

Problem:

Integrating $\int \tan^k x \sec^j x \, dx$ **when** k **is Odd**

Evaluate $\int \tan^5 x \sec^3 x \, dx$.

Solution:

Since the power on $\tan x$ is odd, begin by rewriting $\tan^5 x \sec^3 x = \tan^4 x \sec^2 x \sec x \tan x$. Thus,

Equation:

$$\begin{aligned}
 \tan^5 x \sec^3 x &= \tan^4 x \sec^2 x \sec x \tan x. && \text{Write } \tan^4 x = (\tan^2 x)^2. \\
 \int \tan^5 x \sec^3 x \, dx &= \int (\tan^2 x)^2 \sec^2 x \sec x \tan x \, dx && \text{Use } \tan^2 x = \sec^2 x - 1. \\
 &= \int (\sec^2 x - 1)^2 \sec^2 x \sec x \tan x \, dx && \text{Let } u = \sec x \text{ and } du = \sec x \tan x \, dx. \\
 &= \int (u^2 - 1)^2 u^2 \, du && \text{Expand.} \\
 &= \int (u^6 - 2u^4 + u^2) \, du && \text{Integrate.} \\
 &= \frac{1}{7} u^7 - \frac{2}{5} u^5 + \frac{1}{3} u^3 + C && \text{Substitute } \sec x = u. \\
 &= \frac{1}{7} \sec^7 x - \frac{2}{5} \sec^5 x + \frac{1}{3} \sec^3 x + C.
 \end{aligned}$$

Example:

Exercise:

Problem:

Integrating $\int \tan^k x \, dx$ **where** k **is Odd and** $k \geq 3$

Evaluate $\int \tan^3 x \, dx$.

Solution:

Begin by rewriting $\tan^3 x = \tan x \tan^2 x = \tan x (\sec^2 x - 1) = \tan x \sec^2 x - \tan x$. Thus,

Equation:

$$\begin{aligned}\int \tan^3 x \, dx &= \int (\tan x \sec^2 x - \tan x) \, dx \\ &= \int \tan x \sec^2 x \, dx - \int \tan x \, dx \\ &= \frac{1}{2} \tan^2 x - \ln |\sec x| + C.\end{aligned}$$

For the first integral, use the substitution $u = \tan x$. For the second integral, use the formula.

Example:

Exercise:

Problem:

Integrating $\int \sec^3 x \, dx$

Integrate $\int \sec^3 x \, dx$.

Solution:

This integral requires integration by parts. To begin, let $u = \sec x$ and $dv = \sec^2 x \, dx$. These choices make $du = \sec x \tan x$ and $v = \tan x$. Thus,

Equation:

$$\begin{aligned}\int \sec^3 x \, dx &= \sec x \tan x - \int \tan x \sec x \tan x \, dx \\ &= \sec x \tan x - \int \tan^2 x \sec x \, dx && \text{Simplify.} \\ &= \sec x \tan x - \int (\sec^2 x - 1) \sec x \, dx && \text{Substitute } \tan^2 x = \sec^2 x - 1. \\ &= \sec x \tan x + \int \sec x \, dx - \int \sec^3 x \, dx && \text{Rewrite.} \\ &= \sec x \tan x + \ln |\sec x + \tan x| - \int \sec^3 x \, dx. && \text{Evaluate } \int \sec x \, dx.\end{aligned}$$

We now have

Equation:

$$\int \sec^3 x \, dx = \sec x \tan x + \ln |\sec x + \tan x| - \int \sec^3 x \, dx.$$

Since the integral $\int \sec^3 x \, dx$ has reappeared on the right-hand side, we can solve for $\int \sec^3 x \, dx$ by adding it to both sides. In doing so, we obtain

Equation:

$$2 \int \sec^3 x \, dx = \sec x \tan x + \ln |\sec x + \tan x|.$$

Dividing by 2, we arrive at

Equation:

$$\int \sec^3 x \, dx = \frac{1}{2} \sec x \tan x + \frac{1}{2} \ln |\sec x + \tan x| + C.$$

Note:

Exercise:

Problem: Evaluate $\int \tan^3 x \sec^7 x \, dx$.

Solution:

$$\frac{1}{9} \sec^9 x - \frac{1}{7} \sec^7 x + C$$

Hint

Use [\[link\]](#) as a guide.

Reduction Formulas

Evaluating $\int \sec^n x \, dx$ for values of n where n is odd requires integration by parts. In addition, we must also know the value of $\int \sec^{n-2} x \, dx$ to evaluate $\int \sec^n x \, dx$. The evaluation of $\int \tan^n x \, dx$ also requires being able to integrate $\int \tan^{n-2} x \, dx$. To make the process easier, we can derive and apply the following **power reduction formulas**. These rules allow us to replace the integral of a power of $\sec x$ or $\tan x$ with the integral of a lower power of $\sec x$ or $\tan x$.

Note:

Rule: Reduction Formulas for $\int \sec^n x \, dx$ and $\int \tan^n x \, dx$

Equation:

$$\int \sec^n x \, dx = \frac{1}{n-1} \sec^{n-2} x \tan x + \frac{n-2}{n-1} \int \sec^{n-2} x \, dx$$

Equation:

$$\int \tan^n x \, dx = \frac{1}{n-1} \tan^{n-1} x - \int \tan^{n-2} x \, dx$$

The first power reduction rule may be verified by applying integration by parts. The second may be verified by following the strategy outlined for integrating odd powers of $\tan x$.

Example:

Exercise:

Problem:

Revisiting $\int \sec^3 x \, dx$

Apply a reduction formula to evaluate $\int \sec^3 x \, dx$.

Solution:

By applying the first reduction formula, we obtain

Equation:

$$\begin{aligned} \int \sec^3 x \, dx &= \frac{1}{2} \sec x \tan x + \frac{1}{2} \int \sec x \, dx \\ &= \frac{1}{2} \sec x \tan x + \frac{1}{2} \ln |\sec x + \tan x| + C. \end{aligned}$$

Example:

Exercise:

Problem:

Using a Reduction Formula

Evaluate $\int \tan^4 x \, dx$.

Solution:

Applying the reduction formula for $\int \tan^4 x \, dx$ we have

Equation:

$$\begin{aligned}
 \int \tan^4 x \, dx &= \frac{1}{3} \tan^3 x - \int \tan^2 x \, dx \\
 &= \frac{1}{3} \tan^3 x - (\tan x - \int \tan^0 x \, dx) && \text{Apply the reduction formula to } \int \tan^2 x \, dx. \\
 &= \frac{1}{3} \tan^3 x - \tan x + \int 1 \, dx && \text{Simplify.} \\
 &= \frac{1}{3} \tan^3 x - \tan x + x + C. && \text{Evaluate } \int 1 \, dx.
 \end{aligned}$$

Note:

Exercise:

Problem: Apply the reduction formula to $\int \sec^5 x \, dx$.

Solution:

$$\int \sec^5 x \, dx = \frac{1}{4} \sec^3 x \tan x - \frac{3}{4} \int \sec^3 x$$

Hint

Use reduction formula 1 and let $n = 5$.

Key Concepts

- Integrals of trigonometric functions can be evaluated by the use of various strategies. These strategies include
 - Applying trigonometric identities to rewrite the integral so that it may be evaluated by u -substitution
 - Using integration by parts
 - Applying trigonometric identities to rewrite products of sines and cosines with different arguments as the sum of individual sine and cosine functions
 - Applying reduction formulas

Key Equations

To integrate products involving $\sin(ax)$, $\sin(bx)$, $\cos(ax)$, and $\cos(bx)$, use the substitutions.

- Sine Products**
 $\sin(ax)\sin(bx) = \frac{1}{2}\cos((a-b)x) - \frac{1}{2}\cos((a+b)x)$
- Sine and Cosine Products**
 $\sin(ax)\cos(bx) = \frac{1}{2}\sin((a-b)x) + \frac{1}{2}\sin((a+b)x)$
- Cosine Products**
 $\cos(ax)\cos(bx) = \frac{1}{2}\cos((a-b)x) + \frac{1}{2}\cos((a+b)x)$

- **Power Reduction Formula**

$$\int \sec^n x \, dx = \frac{1}{n-1} \sec^{n-1} x + \frac{n-2}{n-1} \int \sec^{n-2} x \, dx$$

- **Power Reduction Formula**

$$\int \tan^n x \, dx = \frac{1}{n-1} \tan^{n-1} x - \int \tan^{n-2} x \, dx$$

Fill in the blank to make a true statement.

Exercise:

Problem: $\sin^2 x + \underline{\hspace{2cm}} = 1$

Solution:

$$\cos^2 x$$

Exercise:

Problem: $\sec^2 x - 1 = \underline{\hspace{2cm}}$

Use an identity to reduce the power of the trigonometric function to a trigonometric function raised to the first power.

Exercise:

Problem: $\sin^2 x = \underline{\hspace{2cm}}$

Solution:

$$\frac{1 - \cos(2x)}{2}$$

Exercise:

Problem: $\cos^2 x = \underline{\hspace{2cm}}$

Evaluate each of the following integrals by u -substitution.

Exercise:

Problem: $\int \sin^3 x \cos x \, dx$

Solution:

$$\frac{\sin^4 x}{4} + C$$

Exercise:

Problem: $\int \sqrt{\cos x} \sin x \, dx$

Exercise:

Problem: $\int \tan^5(2x) \sec^2(2x) \, dx$

Solution:

$$\frac{1}{12}\tan^6(2x) + C$$

Exercise:

Problem: $\int \sin^7(2x)\cos(2x)dx$

Exercise:

Problem: $\int \tan\left(\frac{x}{2}\right)\sec^2\left(\frac{x}{2}\right)dx$

Solution:

$$\sec^2\left(\frac{x}{2}\right) + C$$

Exercise:

Problem: $\int \tan^2 x \sec^2 x \, dx$

Compute the following integrals using the guidelines for integrating powers of trigonometric functions. Use a CAS to check the solutions. (*Note:* Some of the problems may be done using techniques of integration learned previously.)

Exercise:

Problem: $\int \sin^3 x \, dx$

Solution:

$$-\cos x + \frac{1}{3}\cos^3 x + C$$

Exercise:

Problem: $\int \cos^3 x \, dx$

Exercise:

Problem: $\int \sin x \cos x \, dx$

Solution:

$$-\frac{1}{2}\cos^2 x + C \text{ or } \frac{1}{2}\sin^2 x + C$$

Exercise:

Problem: $\int \cos^5 x \, dx$

Exercise:

Problem: $\int \sin^5 x \cos^2 x \, dx$

Solution:

$$-\frac{1}{3}\cos^3 x + \frac{2}{5}\cos^5 x - \frac{1}{7}\cos^7 x + C$$

Exercise:

Problem: $\int \sin^3 x \cos^3 x \, dx$

Exercise:

Problem: $\int \sqrt{\sin x} \cos x \, dx$

Solution:

$$\frac{2}{3}(\sin x)^{\frac{3}{2}} + C$$

Exercise:

Problem: $\int \sqrt{\sin x} \cos^3 x \, dx$

Exercise:

Problem: $\int \sec x \tan x \, dx$

Solution:

$$\sec x + C$$

Exercise:

Problem: $\int \tan(5x) \, dx$

Exercise:

Problem: $\int \tan^2 x \sec x \, dx$

Solution:

$$\frac{1}{2}\sec x \tan x - \frac{1}{2}\ln|\sec x + \tan x| + C$$

Exercise:

Problem: $\int \tan x \sec^3 x \, dx$

Exercise:

Problem: $\int \sec^4 x \, dx$

Solution:

$$\frac{2\tan x}{3} + \frac{1}{3}\sec(x)^2 \tan x = \tan x + \frac{\tan^3 x}{3} + C$$

Exercise:

Problem: $\int \cot x \, dx$

Exercise:

Problem: $\int \csc x \, dx$

Solution:

$$-\ln |\cot x + \csc x| + C$$

Exercise:

Problem: $\int \frac{\tan^3 x}{\sqrt{\sec x}} \, dx$

For the following exercises, find a general formula for the integrals.

Exercise:

Problem: $\int \sin^2 ax \cos ax \, dx$

Solution:

$$\frac{\sin^3(ax)}{3a} + C$$

Exercise:

Problem: $\int \sin ax \cos ax \, dx.$

Use the double-angle formulas to evaluate the following integrals.

Exercise:

Problem: $\int_0^\pi \sin^2 x \, dx$

Solution:

$$\frac{\pi}{2}$$

Exercise:

Problem: $\int_0^{\pi} \sin^4 x \, dx$

Exercise:

Problem: $\int \cos^2 3x \, dx$

Solution:

$$\frac{x}{2} + \frac{1}{12} \sin(6x) + C$$

Exercise:

Problem: $\int \sin^2 x \cos^2 x \, dx$

Exercise:

Problem: $\int \sin^2 x \, dx + \int \cos^2 x \, dx$

Solution:

$$x + C$$

Exercise:

Problem: $\int \sin^2 x \cos^2(2x) \, dx$

For the following exercises, evaluate the definite integrals. Express answers in exact form whenever possible.

Exercise:

Problem: $\int_0^{2\pi} \cos x \sin 2x \, dx$

Solution:

$$0$$

Exercise:

Problem: $\int_0^{\pi} \sin 3x \sin 5x \, dx$

Exercise:

Problem: $\int_0^{\pi} \cos(99x)\sin(101x)dx$

Solution:

0

Exercise:

Problem: $\int_{-\pi}^{\pi} \cos^2(3x)dx$

Exercise:

Problem: $\int_0^{2\pi} \sin x \sin(2x)\sin(3x)dx$

Solution:

0

Exercise:

Problem: $\int_0^{4\pi} \cos(x/2)\sin(x/2)dx$

Exercise:

Problem: $\int_{\pi/6}^{\pi/3} \frac{\cos^3 x}{\sqrt{\sin x}} dx$ (Round this answer to three decimal places.)

Solution:

Approximately 0.239

Exercise:

Problem: $\int_{-\pi/3}^{\pi/3} \sqrt{\sec^2 x - 1} dx$

Exercise:

Problem: $\int_0^{\pi/2} \sqrt{1 - \cos(2x)} dx$

Solution:

$\sqrt{2}$

Exercise:

Problem:

Find the area of the region bounded by the graphs of the equations $y = \sin x$, $y = \sin^3 x$, $x = 0$, and $x = \frac{\pi}{2}$.

Exercise:**Problem:**

Find the area of the region bounded by the graphs of the equations $y = \cos^2 x$, $y = \sin^2 x$, $x = -\frac{\pi}{4}$, and $x = \frac{\pi}{4}$.

Solution:

1.0

Exercise:**Problem:**

A particle moves in a straight line with the velocity function $v(t) = \sin(\omega t)\cos^2(\omega t)$. Find its position function $x = f(t)$ if $f(0) = 0$.

Exercise:

Problem: Find the average value of the function $f(x) = \sin^2 x \cos^3 x$ over the interval $[-\pi, \pi]$.

Solution:

0

For the following exercises, solve the differential equations.

Exercise:

Problem: $\frac{dy}{dx} = \sin^2 x$. The curve passes through point $(0, 0)$.

Exercise:

Problem: $\frac{dy}{d\theta} = \sin^4(\pi\theta)$

Solution:

$$\frac{3\theta}{8} - \frac{1}{4\pi} \sin(2\pi\theta) + \frac{1}{32\pi} \sin(4\pi\theta) + C = f(x)$$

Exercise:

Problem: Find the length of the curve $y = \ln(\csc x)$, $\frac{\pi}{4} \leq x \leq \frac{\pi}{2}$.

Exercise:

Problem: Find the length of the curve $y = \ln(\sin x)$, $\frac{\pi}{3} \leq x \leq \frac{\pi}{2}$.

Solution:

$$\ln(\sqrt{3})$$

Exercise:

Problem: Find the volume generated by revolving the curve $y = \cos(3x)$ about the x -axis, $0 \leq x \leq \frac{\pi}{36}$.

For the following exercises, use this information: The inner product of two functions f and g over $[a, b]$ is defined by $f(x) \cdot g(x) = \langle f, g \rangle = \int_a^b f \cdot g dx$. Two distinct functions f and g are said to be orthogonal if $\langle f, g \rangle = 0$.

Exercise:

Problem: Show that $\{\sin(2x), \cos(3x)\}$ are orthogonal over the interval $[-\pi, \pi]$.

Solution:

$$\int_{-\pi}^{\pi} \sin(2x)\cos(3x)dx = 0$$

Exercise:

Problem: Evaluate $\int_{-\pi}^{\pi} \sin(mx)\cos(nx)dx$.

Exercise:

Problem: Integrate $y' = \sqrt{\tan x} \sec^4 x$.

Solution:

$$\sqrt{\tan(x)}x\left(\frac{8\tan x}{21} + \frac{2}{7}\sec x^2\tan x\right) + C = f(x)$$

For each pair of integrals, determine which one is more difficult to evaluate. Explain your reasoning.

Exercise:

Problem: $\int \sin^{456} x \cos x \, dx$ or $\int \sin^2 x \cos^2 x \, dx$

Exercise:

Problem: $\int \tan^{350} x \sec^2 x \, dx$ or $\int \tan^{350} x \sec x \, dx$

Solution:

The second integral is more difficult because the first integral is simply a u -substitution type.

Glossary

power reduction formula

a rule that allows an integral of a power of a trigonometric function to be exchanged for an integral involving a lower power

trigonometric integral

an integral involving powers and products of trigonometric functions

Trigonometric Substitution

- Solve integration problems involving the square root of a sum or difference of two squares.

In this section, we explore integrals containing expressions of the form $\sqrt{a^2 - x^2}$, $\sqrt{a^2 + x^2}$, and $\sqrt{x^2 - a^2}$, where the values of a are positive. We have already encountered and evaluated integrals containing some expressions of this type, but many still remain inaccessible. The technique of **trigonometric substitution** comes in very handy when evaluating these integrals. This technique uses substitution to rewrite these integrals as trigonometric integrals.

Integrals Involving $\sqrt{a^2 - x^2}$

Before developing a general strategy for integrals containing $\sqrt{a^2 - x^2}$, consider the integral $\int \sqrt{9 - x^2} dx$.

This integral cannot be evaluated using any of the techniques we have discussed so far. However, if we make the substitution $x = 3\sin\theta$, we have $dx = 3\cos\theta d\theta$. After substituting into the integral, we have

Equation:

$$\int \sqrt{9 - x^2} dx = \int \sqrt{9 - (3\sin\theta)^2} 3\cos\theta d\theta.$$

After simplifying, we have

Equation:

$$\int \sqrt{9 - x^2} dx = \int 9\sqrt{1 - \sin^2\theta} \cos\theta d\theta.$$

Letting $1 - \sin^2\theta = \cos^2\theta$, we now have

Equation:

$$\int \sqrt{9 - x^2} dx = \int 9\sqrt{\cos^2\theta} \cos\theta d\theta.$$

Assuming that $\cos\theta \geq 0$, we have

Equation:

$$\int \sqrt{9 - x^2} dx = \int 9\cos^2\theta d\theta.$$

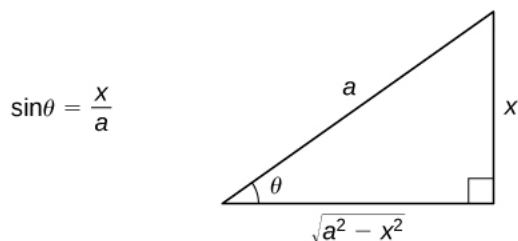
At this point, we can evaluate the integral using the techniques developed for integrating powers and products of trigonometric functions. Before completing this example, let's take a look at the general theory behind this idea.

To evaluate integrals involving $\sqrt{a^2 - x^2}$, we make the substitution $x = a\sin\theta$ and $dx = a\cos\theta$. To see that this actually makes sense, consider the following argument: The domain of $\sqrt{a^2 - x^2}$ is $[-a, a]$. Thus, $-a \leq x \leq a$. Consequently, $-1 \leq \frac{x}{a} \leq 1$. Since the range of $\sin x$ over $[-(\pi/2), \pi/2]$ is $[-1, 1]$, there is a unique angle θ satisfying $-(\pi/2) \leq \theta \leq \pi/2$ so that $\sin\theta = x/a$, or equivalently, so that $x = a\sin\theta$. If we substitute $x = a\sin\theta$ into $\sqrt{a^2 - x^2}$, we get

Equation:

$$\begin{aligned}
 \sqrt{a^2 - x^2} &= \sqrt{a^2 - (a \sin \theta)^2} && \text{Let } x = a \sin \theta \text{ where } -\frac{\pi}{2} \leq \theta \leq \frac{\pi}{2}. \text{ Simplify.} \\
 &= \sqrt{a^2 - a^2 \sin^2 \theta} && \text{Factor out } a^2. \\
 &= \sqrt{a^2(1 - \sin^2 \theta)} && \text{Substitute } 1 - \sin^2 \theta = \cos^2 \theta. \\
 &= \sqrt{a^2 \cos^2 \theta} && \text{Take the square root.} \\
 &= |a \cos \theta| \\
 &= a \cos \theta.
 \end{aligned}$$

Since $\cos \theta \geq 0$ on $-\frac{\pi}{2} \leq \theta \leq \frac{\pi}{2}$ and $a > 0$, $|a \cos \theta| = a \cos \theta$. We can see, from this discussion, that by making the substitution $x = a \sin \theta$, we are able to convert an integral involving a radical into an integral involving trigonometric functions. After we evaluate the integral, we can convert the solution back to an expression involving x . To see how to do this, let's begin by assuming that $0 < x < a$. In this case, $0 < \theta < \frac{\pi}{2}$. Since $\sin \theta = \frac{x}{a}$, we can draw the reference triangle in [\[link\]](#) to assist in expressing the values of $\cos \theta$, $\tan \theta$, and the remaining trigonometric functions in terms of x . It can be shown that this triangle actually produces the correct values of the trigonometric functions evaluated at θ for all θ satisfying $-\frac{\pi}{2} \leq \theta \leq \frac{\pi}{2}$. It is useful to observe that the expression $\sqrt{a^2 - x^2}$ actually appears as the length of one side of the triangle. Last, should θ appear by itself, we use $\theta = \sin^{-1}\left(\frac{x}{a}\right)$.



A reference triangle can help express the trigonometric functions evaluated at θ in terms of x .

The essential part of this discussion is summarized in the following problem-solving strategy.

Note:

Problem-Solving Strategy: Integrating Expressions Involving $\sqrt{a^2 - x^2}$

1. It is a good idea to make sure the integral cannot be evaluated easily in another way. For example, although this method can be applied to integrals of the form $\int \frac{1}{\sqrt{a^2 - x^2}} dx$, $\int \frac{x}{\sqrt{a^2 - x^2}} dx$, and $\int x \sqrt{a^2 - x^2} dx$, they can each be integrated directly either by formula or by a simple u -substitution.
2. Make the substitution $x = a \sin \theta$ and $dx = a \cos \theta d\theta$. *Note:* This substitution yields $\sqrt{a^2 - x^2} = a \cos \theta$.
3. Simplify the expression.
4. Evaluate the integral using techniques from the section on trigonometric integrals.
5. Use the reference triangle from [\[link\]](#) to rewrite the result in terms of x . You may also need to use some trigonometric identities and the relationship $\theta = \sin^{-1}\left(\frac{x}{a}\right)$.

The following example demonstrates the application of this problem-solving strategy.

Example:

Exercise:

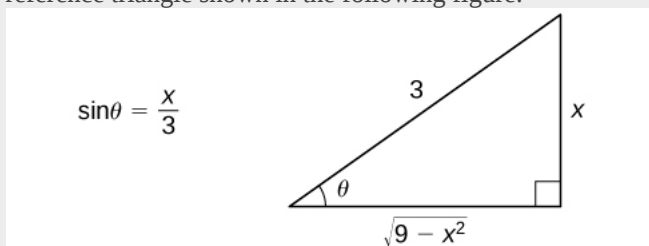
Problem:

Integrating an Expression Involving $\sqrt{a^2 - x^2}$

Evaluate $\int \sqrt{9 - x^2} dx$.

Solution:

Begin by making the substitutions $x = 3 \sin \theta$ and $dx = 3 \cos \theta d\theta$. Since $\sin \theta = \frac{x}{3}$, we can construct the reference triangle shown in the following figure.



A reference triangle can be constructed for [\[link\]](#).

Thus,

Equation:

$$\begin{aligned}
\int \sqrt{9-x^2} dx &= \int \sqrt{9-(3\sin\theta)^2} 3\cos\theta d\theta \\
&= \int \sqrt{9(1-\sin^2\theta)} 3\cos\theta d\theta \\
&= \int \sqrt{9\cos^2\theta} 3\cos\theta d\theta \\
&= \int 3|\cos\theta| 3\cos\theta d\theta \\
&= \int 9\cos^2\theta d\theta \\
&= \int 9\left(\frac{1}{2} + \frac{1}{2}\cos(2\theta)\right) d\theta \\
&= \frac{9}{2}\theta + \frac{9}{4}\sin(2\theta) + C \\
&= \frac{9}{2}\theta + \frac{9}{4}(2\sin\theta\cos\theta) + C \\
&= \frac{9}{2}\sin^{-1}\left(\frac{x}{3}\right) + \frac{9}{2} \cdot \frac{x}{3} \cdot \frac{\sqrt{9-x^2}}{3} + C \\
&= \frac{9}{2}\sin^{-1}\left(\frac{x}{3}\right) + \frac{x\sqrt{9-x^2}}{2} + C.
\end{aligned}$$

Substitute $x = 3\sin\theta$ and $dx = 3\cos\theta d\theta$.

Simplify.

Substitute $\cos^2\theta = 1 - \sin^2\theta$.

Take the square root.

Simplify. Since $-\frac{\pi}{2} \leq \theta \leq \frac{\pi}{2}$, $\cos\theta \geq 0$ and $|\cos\theta| = \cos\theta$.

Use the strategy for integrating an even power of $\cos\theta$.

Evaluate the integral.

Substitute $\sin(2\theta) = 2\sin\theta\cos\theta$.

Substitute $\sin^{-1}\left(\frac{x}{3}\right) = \theta$ and $\sin\theta = \frac{x}{3}$. Use the reference triangle to see that $\cos\theta = \frac{\sqrt{9-x^2}}{3}$ and make this substitution.

Simplify.

Example:

Exercise:

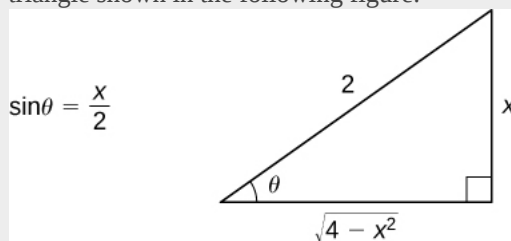
Problem:

Integrating an Expression Involving $\sqrt{a^2 - x^2}$

Evaluate $\int \frac{\sqrt{4-x^2}}{x} dx$.

Solution:

First make the substitutions $x = 2\sin\theta$ and $dx = 2\cos\theta d\theta$. Since $\sin\theta = \frac{x}{2}$, we can construct the reference triangle shown in the following figure.



A reference triangle can be constructed for [\[link\]](#).

Thus,

Equation:

$$\begin{aligned}
 \int \frac{\sqrt{4-x^2}}{x} dx &= \int \frac{\sqrt{4-(2\sin\theta)^2}}{2\sin\theta} 2\cos\theta d\theta \\
 &= \int \frac{2\cos^2\theta}{\sin\theta} d\theta \\
 &= \int \frac{2(1-\sin^2\theta)}{\sin\theta} d\theta \\
 &= \int (2\csc\theta - 2\sin\theta) d\theta \\
 &= 2\ln|\csc\theta - \cot\theta| + 2\cos\theta + C \\
 &= 2\ln\left|\frac{2}{x} - \frac{\sqrt{4-x^2}}{x}\right| + \sqrt{4-x^2} + C.
 \end{aligned}$$

Substitute $x = 2\sin\theta$ and $dx = 2\cos\theta d\theta$.

Substitute $\cos^2\theta = 1 - \sin^2\theta$ and simplify.

Substitute $\sin^2\theta = 1 - \cos^2\theta$.

Separate the numerator, simplify, and use $\csc\theta = \frac{1}{\sin\theta}$.

Evaluate the integral.

Use the reference triangle to rewrite the expression in terms of x and simplify.

In the next example, we see that we sometimes have a choice of methods.

Example:

Exercise:

Problem:

Integrating an Expression Involving $\sqrt{a^2 - x^2}$ Two Ways

Evaluate $\int x^3 \sqrt{1-x^2} dx$ two ways: first by using the substitution $u = 1 - x^2$ and then by using a trigonometric substitution.

Solution:

Method 1

Let $u = 1 - x^2$ and hence $x^2 = 1 - u$. Thus, $du = -2x dx$. In this case, the integral becomes

Equation:

$$\begin{aligned}
 \int x^3 \sqrt{1-x^2} dx &= -\frac{1}{2} \int x^2 \sqrt{1-x^2} (-2x dx) && \text{Make the substitution.} \\
 &= -\frac{1}{2} \int (1-u) \sqrt{u} du && \text{Expand the expression.} \\
 &= -\frac{1}{2} \int (u^{1/2} - u^{3/2}) du && \text{Evaluate the integral.} \\
 &= -\frac{1}{2} \left(\frac{2}{3} u^{3/2} - \frac{2}{5} u^{5/2} \right) + C && \text{Rewrite in terms of } x. \\
 &= -\frac{1}{3} (1-x^2)^{3/2} + \frac{1}{5} (1-x^2)^{5/2} + C.
 \end{aligned}$$

Method 2

Let $x = \sin \theta$. In this case, $dx = \cos \theta d\theta$. Using this substitution, we have

Equation:

$$\begin{aligned}\int x^3 \sqrt{1-x^2} dx &= \int \sin^3 \theta \cos^2 \theta d\theta \\&= \int (1 - \cos^2 \theta) \cos^2 \theta \sin \theta d\theta && \text{Let } u = \cos \theta. \text{ Thus, } du = -\sin \theta d\theta. \\&= \int (u^4 - u^2) du \\&= \frac{1}{5} u^5 - \frac{1}{3} u^3 + C && \text{Substitute } \cos \theta = u. \\&= \frac{1}{5} \cos^5 \theta - \frac{1}{3} \cos^3 \theta + C && \text{Use a reference triangle to see that} \\&= \frac{1}{5} (1 - x^2)^{5/2} - \frac{1}{3} (1 - x^2)^{3/2} + C. && \cos \theta = \sqrt{1 - x^2}.\end{aligned}$$

Note:

Exercise:

Problem:

Rewrite the integral $\int \frac{x^3}{\sqrt{25-x^2}} dx$ using the appropriate trigonometric substitution (do not evaluate the integral).

Solution:

$$\int 125 \sin^3 \theta d\theta$$

Hint

Substitute $x = 5 \sin \theta$ and $dx = 5 \cos \theta d\theta$.

Integrating Expressions Involving $\sqrt{a^2 + x^2}$

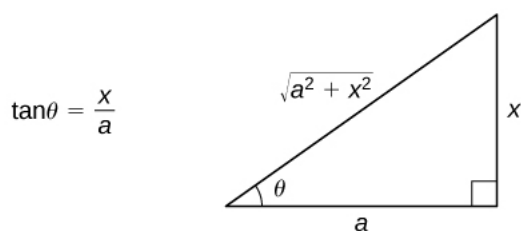
For integrals containing $\sqrt{a^2 + x^2}$, let's first consider the domain of this expression. Since $\sqrt{a^2 + x^2}$ is defined for all real values of x , we restrict our choice to those trigonometric functions that have a range of all real numbers. Thus, our choice is restricted to selecting either $x = a \tan \theta$ or $x = a \cot \theta$. Either of these substitutions would actually work, but the standard substitution is $x = a \tan \theta$ or, equivalently, $\tan \theta = x/a$. With this substitution, we make the assumption that $-(\pi/2) < \theta < \pi/2$, so that we also have $\theta = \tan^{-1}(x/a)$. The procedure for using this substitution is outlined in the following problem-solving strategy.

Note:

Problem-Solving Strategy: Integrating Expressions Involving $\sqrt{a^2 + x^2}$

1. Check to see whether the integral can be evaluated easily by using another method. In some cases, it is more convenient to use an alternative method.
2. Substitute $x = a \tan \theta$ and $dx = a \sec^2 \theta d\theta$. This substitution yields

$$\sqrt{a^2 + x^2} = \sqrt{a^2 + (a \tan \theta)^2} = \sqrt{a^2(1 + \tan^2 \theta)} = \sqrt{a^2 \sec^2 \theta} = |a \sec \theta| = a \sec \theta. \text{ (Since } -\frac{\pi}{2} < \theta < \frac{\pi}{2} \text{ and } \sec \theta > 0 \text{ over this interval, } |a \sec \theta| = a \sec \theta. \text{)}$$
3. Simplify the expression.
4. Evaluate the integral using techniques from the section on trigonometric integrals.
5. Use the reference triangle from [\[link\]](#) to rewrite the result in terms of x . You may also need to use some trigonometric identities and the relationship $\theta = \tan^{-1} \left(\frac{x}{a} \right)$. (Note: The reference triangle is based on the assumption that $x > 0$; however, the trigonometric ratios produced from the reference triangle are the same as the ratios for which $x \leq 0$.)



A reference triangle can be constructed to express the trigonometric functions evaluated at θ in terms of x .

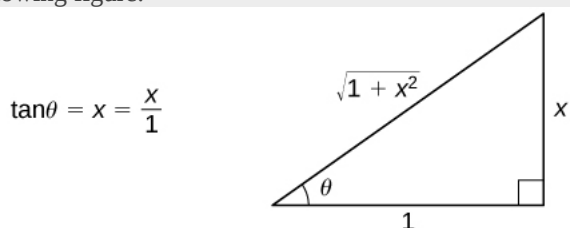
Example:**Exercise:****Problem:**

Integrating an Expression Involving $\sqrt{a^2 + x^2}$

Evaluate $\int \frac{dx}{\sqrt{1 + x^2}}$ and check the solution by differentiating.

Solution:

Begin with the substitution $x = \tan \theta$ and $dx = \sec^2 \theta d\theta$. Since $\tan \theta = x$, draw the reference triangle in the following figure.



The reference triangle for [\[link\]](#).

Thus,

Equation:

$$\begin{aligned}\int \frac{dx}{\sqrt{1+x^2}} &= \int \frac{\sec^2\theta}{\sec\theta} d\theta \\ &= \int \sec\theta d\theta \\ &= \ln|\sec\theta + \tan\theta| + C \\ &= \ln|\sqrt{1+x^2} + x| + C.\end{aligned}$$

Substitute $x = \tan\theta$ and $dx = \sec^2\theta d\theta$. This substitution makes $\sqrt{1+x^2} = \sec\theta$. Simplify.

Evaluate the integral.

Use the reference triangle to express the result in terms of x .

To check the solution, differentiate:

Equation:

$$\begin{aligned}\frac{d}{dx} \left(\ln|\sqrt{1+x^2} + x| \right) &= \frac{1}{\sqrt{1+x^2} + x} \cdot \left(\frac{x}{\sqrt{1+x^2}} + 1 \right) \\ &= \frac{1}{\sqrt{1+x^2} + x} \cdot \frac{x + \sqrt{1+x^2}}{\sqrt{1+x^2}} \\ &= \frac{1}{\sqrt{1+x^2}}.\end{aligned}$$

Since $\sqrt{1+x^2} + x > 0$ for all values of x , we could rewrite $\ln|\sqrt{1+x^2} + x| + C = \ln(\sqrt{1+x^2} + x) + C$, if desired.

Example:

Exercise:

Problem:

Evaluating $\int \frac{dx}{\sqrt{1+x^2}}$ **Using a Different Substitution**

Use the substitution $x = \sinh\theta$ to evaluate $\int \frac{dx}{\sqrt{1+x^2}}$.

Solution:

Because $\sinh\theta$ has a range of all real numbers, and $1 + \sinh^2\theta = \cosh^2\theta$, we may also use the substitution $x = \sinh\theta$ to evaluate this integral. In this case, $dx = \cosh\theta d\theta$. Consequently,

Equation:

$$\begin{aligned}
\int \frac{dx}{\sqrt{1+x^2}} &= \int \frac{\cosh \theta}{\sqrt{1+\sinh^2 \theta}} d\theta \\
&= \int \frac{\cosh \theta}{\sqrt{\cosh^2 \theta}} d\theta \\
&= \int \frac{\cosh \theta}{|\cosh \theta|} d\theta \\
&= \int \frac{\cosh \theta}{\cosh \theta} d\theta \\
&= \int 1 d\theta \\
&= \theta + C \\
&= \sinh^{-1} x + C.
\end{aligned}$$

Substitute $x = \sinh \theta$ and $dx = \cosh \theta d\theta$.

Substitute $1 + \sinh^2 \theta = \cosh^2 \theta$.

$$\sqrt{\cosh^2 \theta} = |\cosh \theta|$$

$|\cosh \theta| = \cosh \theta$ since $\cosh \theta > 0$ for all θ .

Simplify.

Evaluate the integral.

Since $x = \sinh \theta$, we know $\theta = \sinh^{-1} x$.

Analysis

This answer looks quite different from the answer obtained using the substitution $x = \tan \theta$. To see that the solutions are the same, set $y = \sinh^{-1} x$. Thus, $\sinh y = x$. From this equation we obtain:

Equation:

$$\frac{e^y - e^{-y}}{2} = x.$$

After multiplying both sides by $2e^y$ and rewriting, this equation becomes:

Equation:

$$e^{2y} - 2xe^y - 1 = 0.$$

Use the quadratic equation to solve for e^y :

Equation:

$$e^y = \frac{2x \pm \sqrt{4x^2 + 4}}{2}.$$

Simplifying, we have:

Equation:

$$e^y = x \pm \sqrt{x^2 + 1}.$$

Since $x - \sqrt{x^2 + 1} < 0$, it must be the case that $e^y = x + \sqrt{x^2 + 1}$. Thus,

Equation:

$$y = \ln \left(x + \sqrt{x^2 + 1} \right).$$

Last, we obtain

Equation:

$$\sinh^{-1} x = \ln \left(x + \sqrt{x^2 + 1} \right).$$

After we make the final observation that, since $x + \sqrt{x^2 + 1} > 0$,

Equation:

$$\ln(x + \sqrt{x^2 + 1}) = \ln|\sqrt{1 + x^2} + x|,$$

we see that the two different methods produced equivalent solutions.

Example:

Exercise:

Problem:

Finding an Arc Length

Find the length of the curve $y = x^2$ over the interval $[0, \frac{1}{2}]$.

Solution:

Because $\frac{dy}{dx} = 2x$, the arc length is given by

Equation:

$$\int_0^{1/2} \sqrt{1 + (2x)^2} dx = \int_0^{1/2} \sqrt{1 + 4x^2} dx.$$

To evaluate this integral, use the substitution $x = \frac{1}{2}\tan\theta$ and $dx = \frac{1}{2}\sec^2\theta d\theta$. We also need to change the limits of integration. If $x = 0$, then $\theta = 0$ and if $x = \frac{1}{2}$, then $\theta = \frac{\pi}{4}$. Thus,

Equation:

$$\begin{aligned} \int_0^{1/2} \sqrt{1 + 4x^2} dx &= \int_0^{\pi/4} \sqrt{1 + \tan^2\theta} \frac{1}{2} \sec^2\theta d\theta \\ &= \frac{1}{2} \int_0^{\pi/4} \sec^3\theta d\theta \\ &= \frac{1}{2} \left(\frac{1}{2} \sec\theta \tan\theta + \ln|\sec\theta + \tan\theta| \right) \bigg|_0^{\pi/4} \\ &= \frac{1}{4} (\sqrt{2} + \ln(\sqrt{2} + 1)). \end{aligned}$$

After substitution,

$\sqrt{1 + 4x^2} = \sec\theta$. Substitute $1 + \tan^2\theta = \sec^2\theta$ and simplify.

We derived this integral in the previous section.

Evaluate and simplify.

Note:

Exercise:

Problem: Rewrite $\int x^3 \sqrt{x^2 + 4} dx$ by using a substitution involving $\tan\theta$.

Solution:

$$\int 32 \tan^3 \theta \sec^3 \theta d\theta$$

Hint

Use $x = 2 \tan \theta$ and $dx = 2 \sec^2 \theta d\theta$.

Integrating Expressions Involving $\sqrt{x^2 - a^2}$

The domain of the expression $\sqrt{x^2 - a^2}$ is $(-\infty, -a] \cup [a, +\infty)$. Thus, either $x \leq -a$ or $x \geq a$. Hence, $\frac{x}{a} \leq -1$ or $\frac{x}{a} \geq 1$. Since these intervals correspond to the range of $\sec \theta$ on the set $[0, \frac{\pi}{2}) \cup (\frac{\pi}{2}, \pi]$, it makes sense to use the substitution $\sec \theta = \frac{x}{a}$ or, equivalently, $x = a \sec \theta$, where $0 \leq \theta < \frac{\pi}{2}$ or $\frac{\pi}{2} < \theta \leq \pi$. The corresponding substitution for dx is $dx = a \sec \theta \tan \theta d\theta$. The procedure for using this substitution is outlined in the following problem-solving strategy.

Note:

Problem-Solving Strategy: Integrals Involving $\sqrt{x^2 - a^2}$

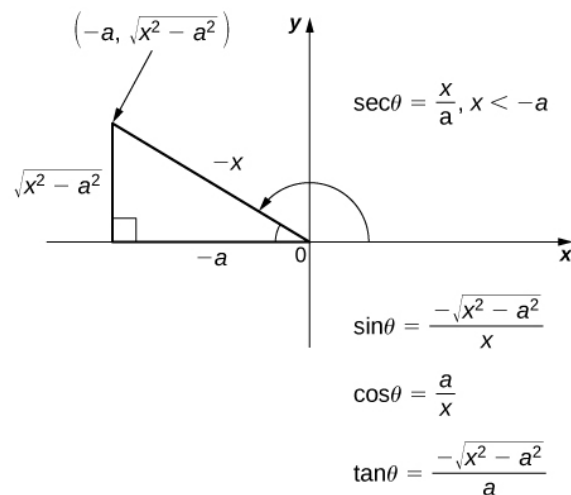
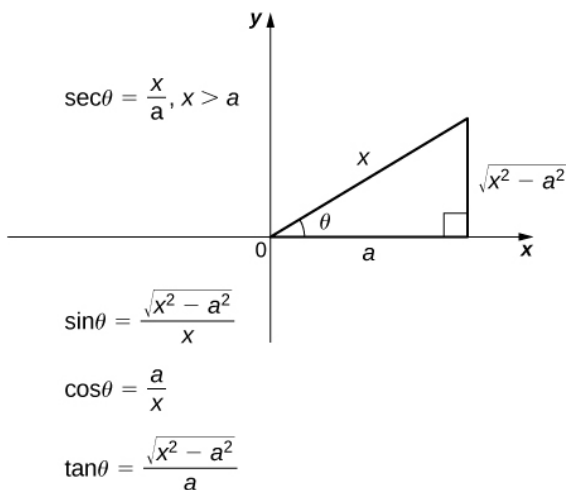
1. Check to see whether the integral cannot be evaluated using another method. If so, we may wish to consider applying an alternative technique.
2. Substitute $x = a \sec \theta$ and $dx = a \sec \theta \tan \theta d\theta$. This substitution yields

Equation:

$$\sqrt{x^2 - a^2} = \sqrt{(a \sec \theta)^2 - a^2} = \sqrt{a^2(\sec^2 \theta - 1)} = \sqrt{a^2 \tan^2 \theta} = |a \tan \theta|.$$

For $x \geq a$, $|a \tan \theta| = a \tan \theta$ and for $x \leq -a$, $|a \tan \theta| = -a \tan \theta$.

3. Simplify the expression.
4. Evaluate the integral using techniques from the section on trigonometric integrals.
5. Use the reference triangles from [\[link\]](#) to rewrite the result in terms of x . You may also need to use some trigonometric identities and the relationship $\theta = \sec^{-1} \left(\frac{x}{a} \right)$. (Note: We need both reference triangles, since the values of some of the trigonometric ratios are different depending on whether $x \geq a$ or $x \leq -a$.)



Use the appropriate reference triangle to express the trigonometric functions evaluated at θ in terms of x .

Example:

Exercise:

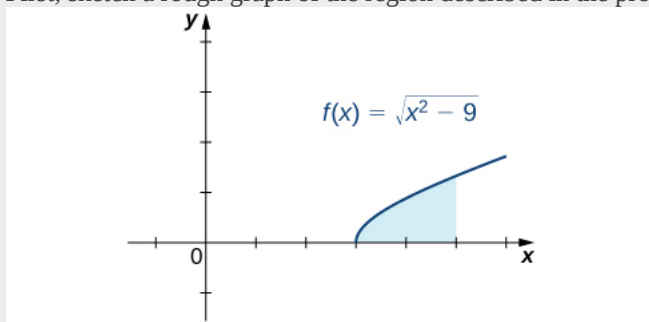
Problem:

Finding the Area of a Region

Find the area of the region between the graph of $f(x) = \sqrt{x^2 - 9}$ and the x -axis over the interval $[3, 5]$.

Solution:

First, sketch a rough graph of the region described in the problem, as shown in the following figure.



Calculating the area of the shaded region requires evaluating an integral with a trigonometric substitution.

We can see that the area is $A = \int_3^5 \sqrt{x^2 - 9} dx$. To evaluate this definite integral, substitute $x = 3\sec\theta$ and $dx = 3\sec\theta\tan\theta d\theta$. We must also change the limits of integration. If $x = 3$, then $3 = 3\sec\theta$ and hence $\theta = 0$. If $x = 5$, then $\theta = \sec^{-1}\left(\frac{5}{3}\right)$. After making these substitutions and simplifying, we have

Equation:

$$\begin{aligned}
\text{Area} &= \int_3^5 \sqrt{x^2 - 9} \, dx \\
&= \int_0^{\sec^{-1}(5/3)} 9 \tan^2 \theta \sec \theta \, d\theta && \text{Use } \tan^2 \theta = 1 - \sec^2 \theta. \\
&= \int_0^{\sec^{-1}(5/3)} 9(\sec^2 \theta - 1) \sec \theta \, d\theta && \text{Expand.} \\
&= \int_0^{\sec^{-1}(5/3)} 9(\sec^3 \theta - \sec \theta) \, d\theta && \text{Evaluate the integral.} \\
&= \left(\frac{9}{2} \ln |\sec \theta + \tan \theta| + \frac{9}{2} \sec \theta \tan \theta \right) - 9 \ln |\sec \theta + \tan \theta| \Bigg|_0^{\sec^{-1}(5/3)} && \text{Simplify.} \\
&= \frac{9}{2} \sec \theta \tan \theta - \frac{9}{2} \ln |\sec \theta + \tan \theta| \Bigg|_0^{\sec^{-1}(5/3)} && \text{Evaluate. Use } \sec(\sec^{-1} \frac{5}{3}) = \frac{5}{3} \\
&= \frac{9}{2} \cdot \frac{5}{3} \cdot \frac{4}{3} - \frac{9}{2} \ln \left| \frac{5}{3} + \frac{4}{3} \right| - \left(\frac{9}{2} \cdot 1 \cdot 0 - \frac{9}{2} \ln |1 + 0| \right) && \text{and } \tan(\sec^{-1} \frac{5}{3}) = \frac{4}{3}. \\
&= 10 - \frac{9}{2} \ln 3.
\end{aligned}$$

Note:

Exercise:

Problem: Evaluate $\int \frac{dx}{\sqrt{x^2 - 4}}$. Assume that $x > 2$.

Solution:

$$\ln \left| \frac{x}{2} + \frac{\sqrt{x^2 - 4}}{2} \right| + C$$

Hint

Substitute $x = 2 \sec \theta$ and $dx = 2 \sec \theta \tan \theta d\theta$.

Key Concepts

- For integrals involving $\sqrt{a^2 - x^2}$, use the substitution $x = a \sin \theta$ and $dx = a \cos \theta d\theta$.
- For integrals involving $\sqrt{a^2 + x^2}$, use the substitution $x = a \tan \theta$ and $dx = a \sec^2 \theta d\theta$.
- For integrals involving $\sqrt{x^2 - a^2}$, substitute $x = a \sec \theta$ and $dx = a \sec \theta \tan \theta d\theta$.

Simplify the following expressions by writing each one using a single trigonometric function.

Exercise:

Problem: $4 - 4 \sin^2 \theta$

Exercise:

Problem: $9 \sec^2 \theta - 9$

Solution:

$$9 \tan^2 \theta$$

Exercise:

Problem: $a^2 + a^2 \tan^2 \theta$

Exercise:

Problem: $a^2 + a^2 \sinh^2 \theta$

Solution:

$$a^2 \cosh^2 \theta$$

Exercise:

Problem: $16 \cosh^2 \theta - 16$

Use the technique of completing the square to express each trinomial as the square of a binomial or the square of a binomial plus a constant.

Exercise:

Problem: $4x^2 - 4x + 1$

Solution:

$$4\left(x - \frac{1}{2}\right)^2$$

Exercise:

Problem: $2x^2 - 8x + 3$

Exercise:

Problem: $-x^2 - 2x + 4$

Solution:

$$-(x + 1)^2 + 5$$

Integrate using the method of trigonometric substitution. Express the final answer in terms of the variable.

Exercise:

Problem: $\int \frac{dx}{\sqrt{4 - x^2}}$

Exercise:

Problem: $\int \frac{dx}{\sqrt{x^2 - a^2}}$

Solution:

$$\ln \left| x + \sqrt{-a^2 + x^2} \right| + C$$

Exercise:

$$\textbf{Problem: } \int \sqrt{4 - x^2} dx$$

Exercise:

$$\textbf{Problem: } \int \frac{dx}{\sqrt{1 + 9x^2}}$$

Solution:

$$\frac{1}{3} \ln \left| \sqrt{9x^2 + 1} + 3x \right| + C$$

Exercise:

$$\textbf{Problem: } \int \frac{x^2 dx}{\sqrt{1 - x^2}}$$

Exercise:

$$\textbf{Problem: } \int \frac{dx}{x^2 \sqrt{1 - x^2}}$$

Solution:

$$-\frac{\sqrt{1-x^2}}{x} + C$$

Exercise:

$$\textbf{Problem: } \int \frac{dx}{(1 + x^2)^2}$$

Exercise:

$$\textbf{Problem: } \int \sqrt{x^2 + 9} dx$$

Solution:

$$9 \left[\frac{x\sqrt{x^2+9}}{18} + \frac{1}{2} \ln \left| \frac{\sqrt{x^2+9}}{3} + \frac{x}{3} \right| \right] + C$$

Exercise:

$$\textbf{Problem: } \int \frac{\sqrt{x^2 - 25}}{x} dx$$

Exercise:

$$\textbf{Problem: } \int \frac{\theta^3 d\theta}{\sqrt{9 - \theta^2}}$$

Solution:

$$-\frac{1}{3}\sqrt{9-\theta^2}(18+\theta^2)+C$$

Exercise:

Problem: $\int \frac{dx}{\sqrt{x^6-x^2}}$

Exercise:

Problem: $\int \sqrt{x^6-x^8}dx$

Solution:

$$\frac{(-1+x^2)(2+3x^2)\sqrt{x^6-x^8}}{15x^3}+C$$

Exercise:

Problem: $\int \frac{dx}{(1+x^2)^{3/2}}$

Exercise:

Problem: $\int \frac{dx}{(x^2-9)^{3/2}}$

Solution:

$$-\frac{x}{9\sqrt{-9+x^2}}+C$$

Exercise:

Problem: $\int \frac{\sqrt{1+x^2}dx}{x}$

Exercise:

Problem: $\int \frac{x^2dx}{\sqrt{x^2-1}}$

Solution:

$$\frac{1}{2}\left(\ln\left|x+\sqrt{x^2-1}\right|+x\sqrt{x^2-1}\right)+C$$

Exercise:

Problem: $\int \frac{x^2dx}{x^2+4}$

Exercise:

Problem: $\int \frac{dx}{x^2 \sqrt{x^2 + 1}}$

Solution:

$$-\frac{\sqrt{1+x^2}}{x} + C$$

Exercise:

Problem: $\int \frac{x^2 dx}{\sqrt{1+x^2}}$

Exercise:

Problem: $\int_{-1}^1 (1-x^2)^{3/2} dx$

Solution:

$$\frac{1}{8} \left(x(5-2x^2)\sqrt{1-x^2} + 3\arcsin x \right) + C$$

In the following exercises, use the substitutions $x = \sinh \theta$, $\cosh \theta$, or $\tanh \theta$. Express the final answers in terms of the variable x .

Exercise:

Problem: $\int \frac{dx}{\sqrt{x^2-1}}$

Exercise:

Problem: $\int \frac{dx}{x\sqrt{1-x^2}}$

Solution:

$$\ln x - \ln |1 + \sqrt{1-x^2}| + C$$

Exercise:

Problem: $\int \sqrt{x^2-1} dx$

Exercise:

Problem: $\int \frac{\sqrt{x^2-1}}{x^2} dx$

Solution:

$$-\frac{\sqrt{-1+x^2}}{x} + \ln |x + \sqrt{-1+x^2}| + C$$

Exercise:

Problem: $\int \frac{dx}{1-x^2}$

Exercise:

Problem: $\int \frac{\sqrt{1+x^2}}{x^2} dx$

Solution:

$$-\frac{\sqrt{1+x^2}}{x} + \operatorname{arcsinh} x + C$$

Use the technique of completing the square to evaluate the following integrals.

Exercise:

Problem: $\int \frac{1}{x^2 - 6x} dx$

Exercise:

Problem: $\int \frac{1}{x^2 + 2x + 1} dx$

Solution:

$$-\frac{1}{1+x} + C$$

Exercise:

Problem: $\int \frac{1}{\sqrt{-x^2 + 2x + 8}} dx$

Exercise:

Problem: $\int \frac{1}{\sqrt{-x^2 + 10x}} dx$

Solution:

$$\frac{2\sqrt{-10+x}\sqrt{x}\ln|\sqrt{-10+x}+\sqrt{x}|}{\sqrt{(10-x)x}} + C$$

Exercise:

Problem: $\int \frac{1}{\sqrt{x^2 + 4x - 12}} dx$

Exercise:

Problem: Evaluate the integral without using calculus: $\int_{-3}^3 \sqrt{9-x^2} dx$.

Solution:

$\frac{9\pi}{2}$; area of a semicircle with radius 3

Exercise:

Problem: Find the area enclosed by the ellipse $\frac{x^2}{4} + \frac{y^2}{9} = 1$.

Exercise:

Problem:

Evaluate the integral $\int \frac{dx}{\sqrt{1-x^2}}$ using two different substitutions. First, let $x = \cos \theta$ and evaluate using trigonometric substitution. Second, let $x = \sin \theta$ and use trigonometric substitution. Are the answers the same?

Solution:

$\arcsin(x) + C$ is the common answer.

Exercise:

Problem:

Evaluate the integral $\int \frac{dx}{x\sqrt{x^2-1}}$ using the substitution $x = \sec \theta$. Next, evaluate the same integral using the substitution $x = \csc \theta$. Show that the results are equivalent.

Exercise:

Problem:

Evaluate the integral $\int \frac{x}{x^2+1} dx$ using the form $\int \frac{1}{u} du$. Next, evaluate the same integral using $x = \tan \theta$. Are the results the same?

Solution:

$\frac{1}{2} \ln(1+x^2) + C$ is the result using either method.

Exercise:

Problem:

State the method of integration you would use to evaluate the integral $\int x\sqrt{x^2+1} dx$. Why did you choose this method?

Exercise:

Problem:

State the method of integration you would use to evaluate the integral $\int x^2\sqrt{x^2-1} dx$. Why did you choose this method?

Solution:

Use trigonometric substitution. Let $x = \sec(\theta)$.

Exercise:

Problem: Evaluate $\int_{-1}^1 \frac{x \, dx}{x^2 + 1}$

Exercise:

Problem:

Find the length of the arc of the curve over the specified interval: $y = \ln x, [1, 5]$. Round the answer to three decimal places.

Solution:

4.367

Exercise:

Problem:

Find the surface area of the solid generated by revolving the region bounded by the graphs of $y = x^2, y = 0, x = 0$, and $x = \sqrt{2}$ about the x -axis. (Round the answer to three decimal places).

Exercise:

Problem:

The region bounded by the graph of $f(x) = \frac{1}{1+x^2}$ and the x -axis between $x = 0$ and $x = 1$ is revolved about the x -axis. Find the volume of the solid that is generated.

Solution:

$$\frac{\pi^2}{8} + \frac{\pi}{4}$$

Solve the initial-value problem for y as a function of x .

Exercise:

Problem: $(x^2 + 36) \frac{dy}{dx} = 1, y(6) = 0$

Exercise:

Problem: $(64 - x^2) \frac{dy}{dx} = 1, y(0) = 3$

Solution:

$$y = \frac{1}{16} \ln \left| \frac{x+8}{x-8} \right| + 3$$

Exercise:

Problem: Find the area bounded by $y = \frac{2}{\sqrt{64-4x^2}}, x = 0, y = 0$, and $x = 2$.

Exercise:

Problem:

An oil storage tank can be described as the volume generated by revolving the area bounded by $y = \frac{16}{\sqrt{64+x^2}}, x = 0, y = 0, x = 2$ about the x -axis. Find the volume of the tank (in cubic meters).

Solution:

$$24.6 \text{ m}^3$$

Exercise:**Problem:**

During each cycle, the velocity v (in feet per second) of a robotic welding device is given by $v = 2t - \frac{14}{4+t^2}$, where t is time in seconds. Find the expression for the displacement s (in feet) as a function of t if $s = 0$ when $t = 0$.

Exercise:

Problem: Find the length of the curve $y = \sqrt{16 - x^2}$ between $x = 0$ and $x = 2$.

Solution:

$$\frac{2\pi}{3}$$

Glossary

trigonometric substitution

an integration technique that converts an algebraic integral containing expressions of the form $\sqrt{a^2 - x^2}$, $\sqrt{a^2 + x^2}$, or $\sqrt{x^2 - a^2}$ into a trigonometric integral

Partial Fractions

- Integrate a rational function using the method of partial fractions.
- Recognize simple linear factors in a rational function.
- Recognize repeated linear factors in a rational function.
- Recognize quadratic factors in a rational function.

We have seen some techniques that allow us to integrate specific rational functions. For example, we know that
Equation:

$$\int \frac{du}{u} = \ln |u| + C \text{ and } \int \frac{du}{u^2 + a^2} = \frac{1}{a} \tan^{-1} \left(\frac{u}{a} \right) + C.$$

However, we do not yet have a technique that allows us to tackle arbitrary quotients of this type. Thus, it is not immediately obvious how to go about evaluating $\int \frac{3x}{x^2 - x - 2} dx$. However, we know from material previously developed that

Equation:

$$\int \left(\frac{1}{x+1} + \frac{2}{x-2} \right) dx = \ln |x+1| + 2 \ln |x-2| + C.$$

In fact, by getting a common denominator, we see that

Equation:

$$\frac{1}{x+1} + \frac{2}{x-2} = \frac{3x}{x^2 - x - 2}.$$

Consequently,

Equation:

$$\int \frac{3x}{x^2 - x - 2} dx = \int \left(\frac{1}{x+1} + \frac{2}{x-2} \right) dx.$$

In this section, we examine the method of **partial fraction decomposition**, which allows us to decompose rational functions into sums of simpler, more easily integrated rational functions. Using this method, we can rewrite an expression such as: $\frac{3x}{x^2 - x - 2}$ as an expression such as $\frac{1}{x+1} + \frac{2}{x-2}$.

The key to the method of partial fraction decomposition is being able to anticipate the form that the decomposition of a rational function will take. As we shall see, this form is both predictable and highly dependent on the factorization of the denominator of the rational function. It is also extremely important to keep in mind that partial fraction decomposition can be applied to a rational function $\frac{P(x)}{Q(x)}$ only if $\deg(P(x)) < \deg(Q(x))$. In the case when $\deg(P(x)) \geq \deg(Q(x))$, we must first perform long division to rewrite the quotient $\frac{P(x)}{Q(x)}$ in the form $A(x) + \frac{R(x)}{Q(x)}$, where $\deg(R(x)) < \deg(Q(x))$. We then do a partial fraction decomposition on $\frac{R(x)}{Q(x)}$. The following example, although not requiring partial fraction decomposition, illustrates our approach to integrals of rational functions of the form $\int \frac{P(x)}{Q(x)} dx$, where $\deg(P(x)) \geq \deg(Q(x))$.

Example:

Exercise:**Problem:**

Integrating $\int \frac{P(x)}{Q(x)} dx$, where $\deg(P(x)) \geq \deg(Q(x))$

Evaluate $\int \frac{x^2 + 3x + 5}{x + 1} dx$.

Solution:

Since $\deg(x^2 + 3x + 5) \geq \deg(x + 1)$, we perform long division to obtain

Equation:

$$\frac{x^2 + 3x + 5}{x + 1} = x + 2 + \frac{3}{x + 1}.$$

Thus,

Equation:

$$\begin{aligned} \int \frac{x^2 + 3x + 5}{x + 1} dx &= \int \left(x + 2 + \frac{3}{x + 1} \right) dx \\ &= \frac{1}{2}x^2 + 2x + 3 \ln|x + 1| + C. \end{aligned}$$

Note:

Visit this [website](#) for a review of long division of polynomials.

Note:**Exercise:**

Problem: Evaluate $\int \frac{x - 3}{x + 2} dx$.

Solution:

$$x - 5 \ln|x + 2| + C$$

Hint

Use long division to obtain $\frac{x-3}{x+2} = 1 - \frac{5}{x+2}$.

To integrate $\int \frac{P(x)}{Q(x)} dx$, where $\deg(P(x)) < \deg(Q(x))$, we must begin by factoring $Q(x)$.

Nonrepeated Linear Factors

If $Q(x)$ can be factored as $(a_1x + b_1)(a_2x + b_2) \dots (a_nx + b_n)$, where each linear factor is distinct, then it is possible to find constants $A_1, A_2, \dots, A_n$ satisfying

Equation:

$$\frac{P(x)}{Q(x)} = \frac{A_1}{a_1x + b_1} + \frac{A_2}{a_2x + b_2} + \dots + \frac{A_n}{a_nx + b_n}.$$

The proof that such constants exist is beyond the scope of this course.

In this next example, we see how to use partial fractions to integrate a rational function of this type.

Example:

Exercise:

Problem:

Partial Fractions with Nonrepeated Linear Factors

Evaluate $\int \frac{3x + 2}{x^3 - x^2 - 2x} dx$.

Solution:

Since $\deg(3x + 2) < \deg(x^3 - x^2 - 2x)$, we begin by factoring the denominator of $\frac{3x+2}{x^3-x^2-2x}$. We can see that $x^3 - x^2 - 2x = x(x - 2)(x + 1)$. Thus, there are constants A , B , and C satisfying

Equation:

$$\frac{3x + 2}{x(x - 2)(x + 1)} = \frac{A}{x} + \frac{B}{x - 2} + \frac{C}{x + 1}.$$

We must now find these constants. To do so, we begin by getting a common denominator on the right. Thus,

Equation:

$$\frac{3x + 2}{x(x - 2)(x + 1)} = \frac{A(x - 2)(x + 1) + Bx(x + 1) + Cx(x - 2)}{x(x - 2)(x + 1)}.$$

Now, we set the numerators equal to each other, obtaining

Equation:

$$3x + 2 = A(x - 2)(x + 1) + Bx(x + 1) + Cx(x - 2).$$

There are two different strategies for finding the coefficients A , B , and C . We refer to these as the *method of equating coefficients* and the *method of strategic substitution*.

Note:

Rule: Method of Equating Coefficients

Rewrite [\[link\]](#) in the form

Equation:

$$3x + 2 = (A + B + C)x^2 + (-A + B - 2C)x + (-2A).$$

Equating coefficients produces the system of equations

Equation:

$$\begin{aligned}A + B + C &= 0 \\ -A + B - 2C &= 3 \\ -2A &= 2.\end{aligned}$$

To solve this system, we first observe that $-2A = 2 \Rightarrow A = -1$. Substituting this value into the first two equations gives us the system

Equation:

$$\begin{aligned}B + C &= 1 \\ B - 2C &= 2.\end{aligned}$$

Multiplying the second equation by -1 and adding the resulting equation to the first produces

Equation:

$$-3C = 1,$$

which in turn implies that $C = -\frac{1}{3}$. Substituting this value into the equation $B + C = 1$ yields $B = \frac{4}{3}$.

Thus, solving these equations yields $A = -1$, $B = \frac{4}{3}$, and $C = -\frac{1}{3}$.

It is important to note that the system produced by this method is consistent if and only if we have set up the decomposition correctly. If the system is inconsistent, there is an error in our decomposition.

Note:

Rule: Method of Strategic Substitution

The method of strategic substitution is based on the assumption that we have set up the decomposition correctly. If the decomposition is set up correctly, then there must be values of A , B , and C that satisfy [\[link\]](#) for *all* values of x . That is, this equation must be true for any value of x we care to substitute into it. Therefore, by choosing values of x carefully and substituting them into the equation, we may find A , B , and C easily. For example, if we substitute $x = 0$, the equation reduces to $2 = A(-2)(1)$. Solving for A yields $A = -1$. Next, by substituting $x = 2$, the equation reduces to $8 = B(2)(3)$, or equivalently $B = 4/3$. Last, we substitute $x = -1$ into the equation and obtain $-1 = C(-1)(-3)$. Solving, we have $C = -\frac{1}{3}$. It is important to keep in mind that if we attempt to use this method with a decomposition that has not been set up correctly, we are still able to find values for the constants, but these constants are meaningless. If we do opt to use the method of strategic substitution, then it is a good idea to check the result by recombining the terms algebraically.

Now that we have the values of A , B , and C , we rewrite the original integral:

Equation:

$$\int \frac{3x+2}{x^3-x^2-2x} dx = \int \left(-\frac{1}{x} + \frac{4}{3} \cdot \frac{1}{(x-2)} - \frac{1}{3} \cdot \frac{1}{(x+1)} \right) dx.$$

Evaluating the integral gives us

Equation:

$$\int \frac{3x+2}{x^3-x^2-2x} dx = -\ln|x| + \frac{4}{3}\ln|x-2| - \frac{1}{3}\ln|x+1| + C.$$

In the next example, we integrate a rational function in which the degree of the numerator is not less than the degree of the denominator.

Example:

Exercise:

Problem:

Dividing before Applying Partial Fractions

Evaluate $\int \frac{x^2 + 3x + 1}{x^2 - 4} dx$.

Solution:

Since $\text{degree}(x^2 + 3x + 1) \geq \text{degree}(x^2 - 4)$, we must perform long division of polynomials. This results in

Equation:

$$\frac{x^2 + 3x + 1}{x^2 - 4} = 1 + \frac{3x + 5}{x^2 - 4}.$$

Next, we perform partial fraction decomposition on $\frac{3x+5}{x^2-4} = \frac{3x+5}{(x+2)(x-2)}$. We have

Equation:

$$\frac{3x + 5}{(x - 2)(x + 2)} = \frac{A}{x - 2} + \frac{B}{x + 2}.$$

Thus,

Equation:

$$3x + 5 = A(x + 2) + B(x - 2).$$

Solving for A and B using either method, we obtain $A = 11/4$ and $B = 1/4$.

Rewriting the original integral, we have

Equation:

$$\int \frac{x^2 + 3x + 1}{x^2 - 4} dx = \int \left(1 + \frac{11}{4} \cdot \frac{1}{x - 2} + \frac{1}{4} \cdot \frac{1}{x + 2} \right) dx.$$

Evaluating the integral produces

Equation:

$$\int \frac{x^2 + 3x + 1}{x^2 - 4} dx = x + \frac{11}{4} \ln |x - 2| + \frac{1}{4} \ln |x + 2| + C.$$

As we see in the next example, it may be possible to apply the technique of partial fraction decomposition to a nonrational function. The trick is to convert the nonrational function to a rational function through a substitution.

Example:**Exercise:****Problem:****Applying Partial Fractions after a Substitution**

Evaluate $\int \frac{\cos x}{\sin^2 x - \sin x} dx$.

Solution:

Let's begin by letting $u = \sin x$. Consequently, $du = \cos x dx$. After making these substitutions, we have

Equation:

$$\int \frac{\cos x}{\sin^2 x - \sin x} dx = \int \frac{du}{u^2 - u} = \int \frac{du}{u(u-1)}.$$

Applying partial fraction decomposition to $1/u(u-1)$ gives $\frac{1}{u(u-1)} = -\frac{1}{u} + \frac{1}{u-1}$.

Thus,

Equation:

$$\begin{aligned} \int \frac{\cos x}{\sin^2 x - \sin x} dx &= -\ln |u| + \ln |u-1| + C \\ &= -\ln |\sin x| + \ln |\sin x - 1| + C. \end{aligned}$$

Note:**Exercise:**

Problem: Evaluate $\int \frac{x+1}{(x+3)(x-2)} dx$.

Solution:

$$\frac{2}{5} \ln |x+3| + \frac{3}{5} \ln |x-2| + C$$

Hint

$$\frac{x+1}{(x+3)(x-2)} = \frac{A}{x+3} + \frac{B}{x-2}$$

Repeated Linear Factors

For some applications, we need to integrate rational expressions that have denominators with repeated linear factors—that is, rational functions with at least one factor of the form $(ax+b)^n$, where n is a positive integer greater than or equal to 2. If the denominator contains the repeated linear factor $(ax+b)^n$, then the decomposition must contain

Equation:

$$\frac{A_1}{ax+b} + \frac{A_2}{(ax+b)^2} + \cdots + \frac{A_n}{(ax+b)^n}.$$

As we see in our next example, the basic technique used for solving for the coefficients is the same, but it requires more algebra to determine the numerators of the partial fractions.

Example:

Exercise:

Problem:

Partial Fractions with Repeated Linear Factors

Evaluate $\int \frac{x-2}{(2x-1)^2(x-1)} dx$.

Solution:

We have $\text{degree}(x-2) < \text{degree}((2x-1)^2(x-1))$, so we can proceed with the decomposition. Since $(2x-1)^2$ is a repeated linear factor, include $\frac{A}{2x-1} + \frac{B}{(2x-1)^2}$ in the decomposition. Thus,

Equation:

$$\frac{x-2}{(2x-1)^2(x-1)} = \frac{A}{2x-1} + \frac{B}{(2x-1)^2} + \frac{C}{x-1}.$$

After getting a common denominator and equating the numerators, we have

Equation:

$$x-2 = A(2x-1)(x-1) + B(x-1) + C(2x-1)^2.$$

We then use the method of equating coefficients to find the values of A , B , and C .

Equation:

$$x-2 = (2A+4C)x^2 + (-3A+B-4C)x + (A-B+C).$$

Equating coefficients yields $2A+4C=0$, $-3A+B-4C=1$, and $A-B+C=-2$. Solving this system yields $A=2$, $B=3$, and $C=-1$.

Alternatively, we can use the method of strategic substitution. In this case, substituting $x=1$ and $x=1/2$ into [\[link\]](#) easily produces the values $B=3$ and $C=-1$. At this point, it may seem that we have run out of good choices for x , however, since we already have values for B and C , we can substitute in these values and choose any value for x not previously used. The value $x=0$ is a good option. In this case, we obtain the equation $-2 = A(-1)(-1) + 3(-1) + (-1)(-1)^2$ or, equivalently, $A=2$.

Now that we have the values for A , B , and C , we rewrite the original integral and evaluate it:

Equation:

$$\begin{aligned} \int \frac{x-2}{(2x-1)^2(x-1)} dx &= \int \left(\frac{2}{2x-1} + \frac{3}{(2x-1)^2} - \frac{1}{x-1} \right) dx \\ &= \ln|2x-1| - \frac{3}{2(2x-1)} - \ln|x-1| + C. \end{aligned}$$

Note:**Exercise:****Problem:**

Set up the partial fraction decomposition for $\int \frac{x+2}{(x+3)^3(x-4)^2} dx$. (Do not solve for the coefficients or complete the integration.)

Solution:

$$\frac{x+2}{(x+3)^3(x-4)^2} = \frac{A}{x+3} + \frac{B}{(x+3)^2} + \frac{C}{(x+3)^3} + \frac{D}{(x-4)} + \frac{E}{(x-4)^2}$$

Hint

Use the problem-solving method of [\[link\]](#) for guidance.

The General Method

Now that we are beginning to get the idea of how the technique of partial fraction decomposition works, let's outline the basic method in the following problem-solving strategy.

Note:**Problem-Solving Strategy: Partial Fraction Decomposition**

To decompose the rational function $P(x)/Q(x)$, use the following steps:

1. Make sure that $\deg(P(x)) < \deg(Q(x))$. If not, perform long division of polynomials.
2. Factor $Q(x)$ into the product of linear and irreducible quadratic factors. An irreducible quadratic is a quadratic that has no real zeros.
3. Assuming that $\deg(P(x)) < \deg(Q(x))$, the factors of $Q(x)$ determine the form of the decomposition of $P(x)/Q(x)$.

- a. If $Q(x)$ can be factored as $(a_1x + b_1)(a_2x + b_2) \dots (a_nx + b_n)$, where each linear factor is distinct, then it is possible to find constants $A_1, A_2, \dots, A_n$ satisfying

Equation:

$$\frac{P(x)}{Q(x)} = \frac{A_1}{a_1x + b_1} + \frac{A_2}{a_2x + b_2} + \dots + \frac{A_n}{a_nx + b_n}.$$

- b. If $Q(x)$ contains the repeated linear factor $(ax + b)^n$, then the decomposition must contain

Equation:

$$\frac{A_1}{ax + b} + \frac{A_2}{(ax + b)^2} + \dots + \frac{A_n}{(ax + b)^n}.$$

- c. For each irreducible quadratic factor $ax^2 + bx + c$ that $Q(x)$ contains, the decomposition must include

Equation:

$$\frac{Ax + B}{ax^2 + bx + c}.$$

d. For each repeated irreducible quadratic factor $(ax^2 + bx + c)^n$, the decomposition must include

Equation:

$$\frac{A_1x + B_1}{ax^2 + bx + c} + \frac{A_2x + B_2}{(ax^2 + bx + c)^2} + \cdots + \frac{A_nx + B_n}{(ax^2 + bx + c)^n}.$$

e. After the appropriate decomposition is determined, solve for the constants.

f. Last, rewrite the integral in its decomposed form and evaluate it using previously developed techniques or integration formulas.

Simple Quadratic Factors

Now let's look at integrating a rational expression in which the denominator contains an irreducible quadratic factor. Recall that the quadratic $ax^2 + bx + c$ is irreducible if $ax^2 + bx + c = 0$ has no real zeros—that is, if $b^2 - 4ac < 0$.

Example:

Exercise:

Problem:

Rational Expressions with an Irreducible Quadratic Factor

Evaluate $\int \frac{2x - 3}{x^3 + x} dx$.

Solution:

Since $\deg(2x - 3) < \deg(x^3 + x)$, factor the denominator and proceed with partial fraction decomposition. Since $x^3 + x = x(x^2 + 1)$ contains the irreducible quadratic factor $x^2 + 1$, include $\frac{Ax+B}{x^2+1}$ as part of the decomposition, along with $\frac{C}{x}$ for the linear term x . Thus, the decomposition has the form

Equation:

$$\frac{2x - 3}{x(x^2 + 1)} = \frac{Ax + B}{x^2 + 1} + \frac{C}{x}.$$

After getting a common denominator and equating the numerators, we obtain the equation

Equation:

$$2x - 3 = (Ax + B)x + C(x^2 + 1).$$

Solving for A , B , and C , we get $A = 3$, $B = 2$, and $C = -3$.

Thus,

Equation:

$$\frac{2x - 3}{x^3 + x} = \frac{3x + 2}{x^2 + 1} - \frac{3}{x}.$$

Substituting back into the integral, we obtain

Equation:

$$\begin{aligned}\int \frac{2x-3}{x^3+x} dx &= \int \left(\frac{3x+2}{x^2+1} - \frac{3}{x} \right) dx \\ &= 3 \int \frac{x}{x^2+1} dx + 2 \int \frac{1}{x^2+1} dx - 3 \int \frac{1}{x} dx && \text{Split up the integral.} \\ &= \frac{3}{2} \ln |x^2+1| + 2 \tan^{-1} x - 3 \ln |x| + C. && \text{Evaluate each integral.}\end{aligned}$$

Note: We may rewrite $\ln |x^2+1| = \ln(x^2+1)$, if we wish to do so, since $x^2+1 > 0$.

Example:

Exercise:

Problem:

Partial Fractions with an Irreducible Quadratic Factor

Evaluate $\int \frac{dx}{x^3-8}$.

Solution:

We can start by factoring $x^3-8 = (x-2)(x^2+2x+4)$. We see that the quadratic factor x^2+2x+4 is irreducible since $2^2-4(1)(4) = -12 < 0$. Using the decomposition described in the problem-solving strategy, we get

Equation:

$$\frac{1}{(x-2)(x^2+2x+4)} = \frac{A}{x-2} + \frac{Bx+C}{x^2+2x+4}.$$

After obtaining a common denominator and equating the numerators, this becomes

Equation:

$$1 = A(x^2+2x+4) + (Bx+C)(x-2).$$

Applying either method, we get $A = \frac{1}{12}$, $B = -\frac{1}{12}$, and $C = -\frac{1}{3}$.

Rewriting $\int \frac{dx}{x^3-8}$, we have

Equation:

$$\int \frac{dx}{x^3-8} = \frac{1}{12} \int \frac{1}{x-2} dx - \frac{1}{12} \int \frac{x+4}{x^2+2x+4} dx.$$

We can see that

$\int \frac{1}{x-2} dx = \ln |x-2| + C$, but $\int \frac{x+4}{x^2+2x+4} dx$ requires a bit more effort. Let's begin by completing the square on x^2+2x+4 to obtain

Equation:

$$x^2 + 2x + 4 = (x + 1)^2 + 3.$$

By letting $u = x + 1$ and consequently $du = dx$, we see that

Equation:

$$\begin{aligned}\int \frac{x + 4}{x^2 + 2x + 4} dx &= \int \frac{x + 4}{(x + 1)^2 + 3} dx \\&= \int \frac{u + 3}{u^2 + 3} du \\&= \int \frac{u}{u^2 + 3} du + \int \frac{3}{u^2 + 3} du \\&= \frac{1}{2} \ln |u^2 + 3| + \frac{3}{\sqrt{3}} \tan^{-1} \frac{u}{\sqrt{3}} + C \\&= \frac{1}{2} \ln |x^2 + 2x + 4| + \sqrt{3} \tan^{-1} \left(\frac{x+1}{\sqrt{3}} \right) + C.\end{aligned}$$

Complete the square on the denominator.

Substitute $u = x + 1$, $x = u - 1$, and $du = dx$.

Split the numerator apart.

Evaluate each integral.

Rewrite in terms of x and simplify.

Substituting back into the original integral and simplifying gives

Equation:

$$\int \frac{dx}{x^3 - 8} = \frac{1}{12} \ln |x - 2| - \frac{1}{24} \ln |x^2 + 2x + 4| - \frac{\sqrt{3}}{12} \tan^{-1} \left(\frac{x + 1}{\sqrt{3}} \right) + C.$$

Here again, we can drop the absolute value if we wish to do so, since $x^2 + 2x + 4 > 0$ for all x .

Example:

Exercise:

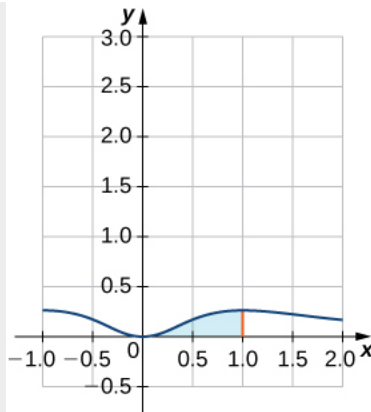
Problem:

Finding a Volume

Find the volume of the solid of revolution obtained by revolving the region enclosed by the graph of $f(x) = \frac{x^2}{(x^2+1)^2}$ and the x -axis over the interval $[0, 1]$ about the y -axis.

Solution:

Let's begin by sketching the region to be revolved (see [\[link\]](#)). From the sketch, we see that the shell method is a good choice for solving this problem.



We can use the shell method to find the volume of revolution obtained by revolving the region shown about the y -axis.

The volume is given by

Equation:

$$V = 2\pi \int_0^1 x \cdot \frac{x^2}{(x^2 + 1)^2} dx = 2\pi \int_0^1 \frac{x^3}{(x^2 + 1)^2} dx.$$

Since $\deg((x^2 + 1)^2) = 4 > 3 = \deg(x^3)$, we can proceed with partial fraction decomposition. Note that $(x^2 + 1)^2$ is a repeated irreducible quadratic. Using the decomposition described in the problem-solving strategy, we get

Equation:

$$\frac{x^3}{(x^2 + 1)^2} = \frac{Ax + B}{x^2 + 1} + \frac{Cx + D}{(x^2 + 1)^2}.$$

Finding a common denominator and equating the numerators gives

Equation:

$$x^3 = (Ax + B)(x^2 + 1) + Cx + D.$$

Solving, we obtain $A = 1$, $B = 0$, $C = -1$, and $D = 0$. Substituting back into the integral, we have

Equation:

$$\begin{aligned}
 V &= 2\pi \int_0^1 \frac{x^3}{(x^2 + 1)^2} dx \\
 &= 2\pi \int_0^1 \left(\frac{x}{x^2 + 1} - \frac{x}{(x^2 + 1)^2} \right) dx \\
 &= 2\pi \left(\frac{1}{2} \ln(x^2 + 1) + \frac{1}{2} \cdot \frac{1}{x^2 + 1} \right) \Bigg|_0^1 \\
 &= \pi \left(\ln 2 - \frac{1}{2} \right).
 \end{aligned}$$

Note:

Exercise:

Problem: Set up the partial fraction decomposition for $\int \frac{x^2 + 3x + 1}{(x + 2)(x - 3)^2(x^2 + 4)^2} dx$.

Solution:

$$\frac{x^2 + 3x + 1}{(x + 2)(x - 3)^2(x^2 + 4)^2} = \frac{A}{x + 2} + \frac{B}{x - 3} + \frac{C}{(x - 3)^2} + \frac{Dx + E}{x^2 + 4} + \frac{Fx + G}{(x^2 + 4)^2}$$

Hint

Use the problem-solving strategy.

Key Concepts

- Partial fraction decomposition is a technique used to break down a rational function into a sum of simple rational functions that can be integrated using previously learned techniques.
- When applying partial fraction decomposition, we must make sure that the degree of the numerator is less than the degree of the denominator. If not, we need to perform long division before attempting partial fraction decomposition.
- The form the decomposition takes depends on the type of factors in the denominator. The types of factors include nonrepeated linear factors, repeated linear factors, nonrepeated irreducible quadratic factors, and repeated irreducible quadratic factors.

Express the rational function as a sum or difference of two simpler rational expressions.

Exercise:

Problem: $\frac{1}{(x - 3)(x - 2)}$

Exercise:

Problem: $\frac{x^2 + 1}{x(x + 1)(x + 2)}$

Solution:

$$-\frac{2}{x+1} + \frac{5}{2(x+2)} + \frac{1}{2x}$$

Exercise:

Problem: $\frac{1}{x^3-x}$

Exercise:

Problem: $\frac{3x+1}{x^2}$

Solution:

$$\frac{1}{x^2} + \frac{3}{x}$$

Exercise:

Problem: $\frac{3x^2}{x^2+1}$ (*Hint: Use long division first.*)

Exercise:

Problem: $\frac{2x^4}{x^2-2x}$

Solution:

$$2x^2 + 4x + 8 + \frac{16}{x-2}$$

Exercise:

Problem: $\frac{1}{(x-1)(x^2+1)}$

Exercise:

Problem: $\frac{1}{x^2(x-1)}$

Solution:

$$-\frac{1}{x^2} - \frac{1}{x} + \frac{1}{x-1}$$

Exercise:

Problem: $\frac{x}{x^2-4}$

Exercise:

Problem: $\frac{1}{x(x-1)(x-2)(x-3)}$

Solution:

$$-\frac{1}{2(x-2)} + \frac{1}{2(x-1)} - \frac{1}{6x} + \frac{1}{6(x-3)}$$

Exercise:

Problem: $\frac{1}{x^4-1} = \frac{1}{(x+1)(x-1)(x^2+1)}$

Exercise:

Problem: $\frac{3x^2}{x^3-1} = \frac{3x^2}{(x-1)(x^2+x+1)}$

Solution:

$$\frac{1}{x-1} + \frac{2x+1}{x^2+x+1}$$

Exercise:

Problem: $\frac{2x}{(x+2)^2}$

Exercise:

Problem: $\frac{3x^4+x^3+20x^2+3x+31}{(x+1)(x^2+4)^2}$

Solution:

$$\frac{2}{x+1} + \frac{x}{x^2+4} - \frac{1}{(x^2+4)^2}$$

Use the method of partial fractions to evaluate each of the following integrals.

Exercise:

Problem: $\int \frac{dx}{(x-3)(x-2)}$

Exercise:

Problem: $\int \frac{3x}{x^2+2x-8} dx$

Solution:

$$-\ln|2-x| + 2\ln|4+x| + C$$

Exercise:

Problem: $\int \frac{dx}{x^3-x}$

Exercise:

Problem: $\int \frac{x}{x^2-4} dx$

Solution:

$$\frac{1}{2}\ln|4-x^2| + C$$

Exercise:

Problem: $\int \frac{dx}{x(x-1)(x-2)(x-3)}$

Exercise:

Problem: $\int \frac{2x^2 + 4x + 22}{x^2 + 2x + 10} dx$

Solution:

$$2 \left(x + \frac{1}{3} \arctan \left(\frac{1+x}{3} \right) \right) + C$$

Exercise:

Problem: $\int \frac{dx}{x^2 - 5x + 6}$

Exercise:

Problem: $\int \frac{2-x}{x^2+x} dx$

Solution:

$$2 \ln |x| - 3 \ln |1+x| + C$$

Exercise:

Problem: $\int \frac{2}{x^2 - x - 6} dx$

Exercise:

Problem: $\int \frac{dx}{x^3 - 2x^2 - 4x + 8}$

Solution:

$$\frac{1}{16} \left(-\frac{4}{-2+x} - \ln |-2+x| + \ln |2+x| \right) + C$$

Exercise:

Problem: $\int \frac{dx}{x^4 - 10x^2 + 9}$

Evaluate the following integrals, which have irreducible quadratic factors.

Exercise:

Problem: $\int \frac{2}{(x-4)(x^2+2x+6)} dx$

Solution:

$$\frac{1}{30} \left(-2\sqrt{5} \arctan \left[\frac{1+x}{\sqrt{5}} \right] + 2 \ln |-4+x| - \ln |6+2x+x^2| \right) + C$$

Exercise:

Problem: $\int \frac{x^2}{x^3 - x^2 + 4x - 4} dx$

Exercise:

Problem: $\int \frac{x^3 + 6x^2 + 3x + 6}{x^3 + 2x^2} dx$

Solution:

$$-\frac{3}{x} + 4 \ln |x + 2| + x + C$$

Exercise:

Problem: $\int \frac{x}{(x-1)(x^2+2x+2)^2} dx$

Use the method of partial fractions to evaluate the following integrals.

Exercise:

Problem: $\int \frac{3x+4}{(x^2+4)(3-x)} dx$

Solution:

$$-\ln |3-x| + \frac{1}{2} \ln |x^2+4| + C$$

Exercise:

Problem: $\int \frac{2}{(x+2)^2(2-x)} dx$

Exercise:

Problem: $\int \frac{3x+4}{x^3-2x-4} dx$ (*Hint: Use the rational root theorem.*)

Solution:

$$\ln |x-2| - \frac{1}{2} \ln |x^2+2x+2| + C$$

Use substitution to convert the integrals to integrals of rational functions. Then use partial fractions to evaluate the integrals.

Exercise:

Problem:

$\int_0^1 \frac{e^x}{36 - e^{2x}} dx$ (Give the exact answer and the decimal equivalent. Round to five decimal places.)

Exercise:

Problem: $\int \frac{e^x dx}{e^{2x} - e^x}$

Solution:

$$-x + \ln|1 - e^x| + C$$

Exercise:

$$\textbf{Problem: } \int \frac{\sin x \, dx}{1 - \cos^2 x}$$

Exercise:

$$\textbf{Problem: } \int \frac{\sin x}{\cos^2 x + \cos x - 6} dx$$

Solution:

$$\frac{1}{5} \ln \left| \frac{\cos x + 3}{\cos x - 2} \right| + C$$

Exercise:

$$\textbf{Problem: } \int \frac{1 - \sqrt{x}}{1 + \sqrt{x}} dx$$

Exercise:

$$\textbf{Problem: } \int \frac{dt}{(e^t - e^{-t})^2}$$

Solution:

$$\frac{1}{2 - 2e^{2t}} + C$$

Exercise:

$$\textbf{Problem: } \int \frac{1 + e^x}{1 - e^x} dx$$

Exercise:

$$\textbf{Problem: } \int \frac{dx}{1 + \sqrt{x+1}}$$

Solution:

$$2\sqrt{1+x} - 2\ln|1 + \sqrt{1+x}| + C$$

Exercise:

$$\textbf{Problem: } \int \frac{dx}{\sqrt{x} + \sqrt[4]{x}}$$

Exercise:

$$\textbf{Problem: } \int \frac{\cos x}{\sin x(1 - \sin x)} dx$$

Solution:

$$\ln \left| \frac{\sin x}{1 - \sin x} \right| + C$$

Exercise:

Problem: $\int \frac{e^x}{(e^{2x} - 4)^2} dx$

Exercise:

Problem: $\int_1^2 \frac{1}{x^2 \sqrt{4 - x^2}} dx$

Solution:

$$\frac{\sqrt{3}}{4}$$

Exercise:

Problem: $\int \frac{1}{2 + e^{-x}} dx$

Exercise:

Problem: $\int \frac{1}{1 + e^x} dx$

Solution:

$$x - \ln(1 + e^x) + C$$

Use the given substitution to convert the integral to an integral of a rational function, then evaluate.

Exercise:

Problem: $\int \frac{1}{t - \sqrt[3]{t}} dt, t = x^3$

Exercise:

Problem: $\int \frac{1}{\sqrt{x} + \sqrt[3]{x}} dx; x = u^6$

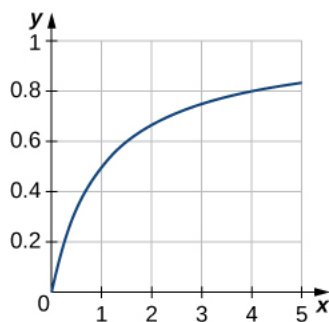
Solution:

$$6x^{1/6} - 3x^{1/3} + 2\sqrt{x} - 6\ln(1 + x^{1/6}) + C$$

Exercise:

Problem:

Graph the curve $y = \frac{x}{1+x}$ over the interval $[0, 5]$. Then, find the area of the region bounded by the curve, the x -axis, and the line $x = 4$.



Exercise:

Problem:

Find the volume of the solid generated when the region bounded by $y = 1/\sqrt{x(3-x)}$, $y = 0$, $x = 1$, and $x = 2$ is revolved about the x -axis.

Solution:

$$\frac{4}{3}\pi \operatorname{arctanh}\left[\frac{1}{3}\right] = \frac{1}{3}\pi \ln 4$$

Exercise:

Problem:

The velocity of a particle moving along a line is a function of time given by $v(t) = \frac{88t^2}{t^2+1}$. Find the distance that the particle has traveled after $t = 5$ sec.

Solve the initial-value problem for x as a function of t .

Exercise:

Problem: $(t^2 - 7t + 12)\frac{dx}{dt} = 1, (t > 4, x(5) = 0)$

Solution:

$$x = -\ln|t-3| + \ln|t-4| + \ln 2$$

Exercise:

Problem: $(t+5)\frac{dx}{dt} = x^2 + 1, t > -5, x(1) = \tan 1$

Exercise:

Problem: $(2t^3 - 2t^2 + t - 1)\frac{dx}{dt} = 3, x(2) = 0$

Solution:

$$x = \ln|t-1| - \sqrt{2}\arctan\left(\sqrt{2}t\right) - \frac{1}{2}\ln\left(t^2 + \frac{1}{2}\right) + \sqrt{2}\arctan\left(2\sqrt{2}\right) + \frac{1}{2}\ln 4.5$$

Exercise:

Problem: Find the x -coordinate of the centroid of the area bounded by

$y(x^2 - 9) = 1, y = 0, x = 4$, and $x = 5$. (Round the answer to two decimal places.)

Exercise:**Problem:**

Find the volume generated by revolving the area bounded by $y = \frac{1}{x^3+7x^2+6x}$, $x = 1$, $x = 7$, and $y = 0$ about the y -axis.

Solution:

$$\frac{2}{5}\pi \ln \frac{28}{13}$$

Exercise:**Problem:**

Find the area bounded by $y = \frac{x-12}{x^2-8x-20}$, $y = 0$, $x = 2$, and $x = 4$. (Round the answer to the nearest hundredth.)

Exercise:

Problem: Evaluate the integral $\int \frac{dx}{x^3+1}$.

Solution:

$$\frac{\arctan\left[\frac{-1+2x}{\sqrt{3}}\right]}{\sqrt{3}} + \frac{1}{3}\ln|1+x| - \frac{1}{6}\ln|1-x+x^2| + C$$

For the following problems, use the substitutions $\tan\left(\frac{x}{2}\right) = t$, $dx = \frac{2}{1+t^2}dt$, $\sin x = \frac{2t}{1+t^2}$, and $\cos x = \frac{1-t^2}{1+t^2}$.

Exercise:

Problem: $\int \frac{dx}{3-5\sin x}$

Exercise:**Problem:**

Find the area under the curve $y = \frac{1}{1+\sin x}$ between $x = 0$ and $x = \pi$. (Assume the dimensions are in inches.)

Solution:

$$2.0 \text{ in.}^2$$

Exercise:

Problem: Given $\tan\left(\frac{x}{2}\right) = t$, derive the formulas $dx = \frac{2}{1+t^2}dt$, $\sin x = \frac{2t}{1+t^2}$, and $\cos x = \frac{1-t^2}{1+t^2}$.

Exercise:

Problem: Evaluate $\int \frac{\sqrt[3]{x-8}}{x} dx$.

Solution:

$$\begin{aligned}
& 3(-8+x)^{1/3} \\
& -2\sqrt{3}\arctan\left[\frac{-1+(-8+x)^{1/3}}{\sqrt{3}}\right] \\
& -2\ln\left[2+(-8+x)^{1/3}\right] \\
& +\ln\left[4-2(-8+x)^{1/3}+(-8+x)^{2/3}\right]+C
\end{aligned}$$

Glossary

partial fraction decomposition

a technique used to break down a rational function into the sum of simple rational functions

Other Strategies for Integration

- Use a table of integrals to solve integration problems.
- Use a computer algebra system (CAS) to solve integration problems.

In addition to the techniques of integration we have already seen, several other tools are widely available to assist with the process of integration. Among these tools are **integration tables**, which are readily available in many books, including the appendices to this one. Also widely available are **computer algebra systems (CAS)**, which are found on calculators and in many campus computer labs, and are free online.

Tables of Integrals

Integration tables, if used in the right manner, can be a handy way either to evaluate or check an integral quickly. Keep in mind that when using a table to check an answer, it is possible for two completely correct solutions to look very different. For example, in [Trigonometric Substitution](#), we found that, by using the substitution $x = \tan \theta$, we can arrive at

Equation:

$$\int \frac{dx}{\sqrt{1+x^2}} = \ln(x + \sqrt{x^2 + 1}) + C.$$

However, using $x = \sinh \theta$, we obtained a different solution—namely,

Equation:

$$\int \frac{dx}{\sqrt{1+x^2}} = \sinh^{-1} x + C.$$

We later showed algebraically that the two solutions are equivalent. That is, we showed that $\sinh^{-1} x = \ln(x + \sqrt{x^2 + 1})$. In this case, the two antiderivatives that we found were actually equal. This need not be the case. However, as long as the difference in the two antiderivatives is a constant, they are equivalent.

Example:

Exercise:

Problem:

Using a Formula from a Table to Evaluate an Integral

Use the table formula

Equation:

$$\int \frac{\sqrt{a^2 - u^2}}{u^2} du = -\frac{\sqrt{a^2 - u^2}}{u} - \sin^{-1} \frac{u}{a} + C$$

to evaluate $\int \frac{\sqrt{16 - e^{2x}}}{e^x} dx$.

Solution:

If we look at integration tables, we see that several formulas contain expressions of the form $\sqrt{a^2 - u^2}$. This expression is actually similar to $\sqrt{16 - e^{2x}}$, where $a = 4$ and $u = e^x$. Keep in mind that we must also have $du = e^x dx$. Multiplying the numerator and the denominator of the given integral by e^x should help to put this integral in a useful form. Thus, we now have

Equation:

$$\int \frac{\sqrt{16 - e^{2x}}}{e^x} dx = \int \frac{\sqrt{16 - e^{2x}}}{e^{2x}} e^x dx.$$

Substituting $u = e^x$ and $du = e^x dx$ produces $\int \frac{\sqrt{a^2 - u^2}}{u^2} du$. From the integration table (#88 in [Appendix A](#)),

Equation:

$$\int \frac{\sqrt{a^2 - u^2}}{u^2} du = -\frac{\sqrt{a^2 - u^2}}{u} - \sin^{-1} \frac{u}{a} + C.$$

Thus,

Equation:

$\int \frac{\sqrt{16 - e^{2x}}}{e^x} dx$	$= \int \frac{\sqrt{16 - e^{2x}}}{e^{2x}} e^x dx$	Substitute $u = e^x$ and $du = e^x dx$.
	$= \int \frac{\sqrt{4^2 - u^2}}{u^2} du$	Apply the formula using $a = 4$.
	$= -\frac{\sqrt{4^2 - u^2}}{u} - \sin^{-1} \frac{u}{4} + C$	Substitute $u = e^x$.
	$= -\frac{\sqrt{16 - e^{2x}}}{u} - \sin^{-1} \left(\frac{e^x}{4} \right) + C.$	

Computer Algebra Systems

If available, a CAS is a faster alternative to a table for solving an integration problem. Many such systems are widely available and are, in general, quite easy to use.

Example:

Exercise:**Problem:****Using a Computer Algebra System to Evaluate an Integral**

Use a computer algebra system to evaluate $\int \frac{dx}{\sqrt{x^2 - 4}}$. Compare this result with $\ln \left| \frac{\sqrt{x^2 - 4}}{2} + \frac{x}{2} \right| + C$, a result we might have obtained if we had used trigonometric substitution.

Solution:

Using Wolfram Alpha, we obtain

Equation:

$$\int \frac{dx}{\sqrt{x^2 - 4}} = \ln \left| \sqrt{x^2 - 4} + x \right| + C.$$

Notice that

Equation:

$$\ln \left| \frac{\sqrt{x^2 - 4}}{2} + \frac{x}{2} \right| + C = \ln \left| \frac{\sqrt{x^2 - 4} + x}{2} \right| + C = \ln \left| \sqrt{x^2 - 4} + x \right| - \ln 2 + C.$$

Since these two antiderivatives differ by only a constant, the solutions are equivalent. We could have also demonstrated that each of these antiderivatives is correct by differentiating them.

Note:

You can access an [integral calculator](#) for more examples.

Example:**Exercise:****Problem:****Using a CAS to Evaluate an Integral**

Evaluate $\int \sin^3 x \, dx$ using a CAS. Compare the result to $\frac{1}{3} \cos^3 x - \cos x + C$, the result we might have obtained using the technique for integrating odd powers of $\sin x$ discussed earlier in this chapter.

Solution:

Using Wolfram Alpha, we obtain

Equation:

$$\int \sin^3 x \, dx = \frac{1}{12}(\cos(3x) - 9\cos x) + C.$$

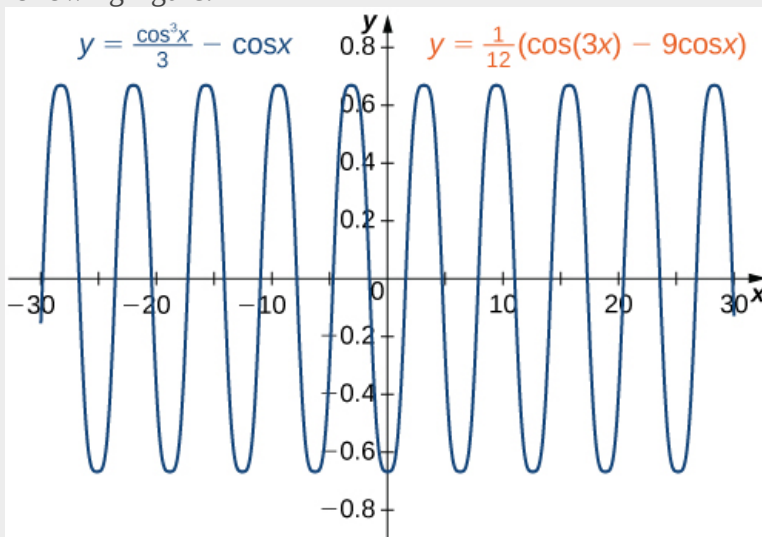
This looks quite different from $\frac{1}{3}\cos^3 x - \cos x + C$. To see that these antiderivatives are equivalent, we can make use of a few trigonometric identities:

Equation:

$$\begin{aligned} \frac{1}{12}(\cos(3x) - 9\cos x) &= \frac{1}{12}(\cos(x + 2x) - 9\cos x) \\ &= \frac{1}{12}(\cos(x)\cos(2x) - \sin(x)\sin(2x) - 9\cos x) \\ &= \frac{1}{12}(\cos x(2\cos^2 x - 1) - \sin x(2\sin x \cos x) - 9\cos x) \\ &= \frac{1}{12}(2\cos^3 x - \cos x - 2\cos x(1 - \cos^2 x) - 9\cos x) \\ &= \frac{1}{12}(4\cos^3 x - 12\cos x) \\ &= \frac{1}{3}\cos^3 x - \cos x. \end{aligned}$$

Thus, the two antiderivatives are identical.

We may also use a CAS to compare the graphs of the two functions, as shown in the following figure.



The graphs of $y = \frac{1}{3}\cos^3 x - \cos x$ and $y = \frac{1}{12}(\cos(3x) - 9\cos x)$ are identical.

Note:

Exercise:

Problem: Use a CAS to evaluate $\int \frac{dx}{\sqrt{x^2 + 4}}$.

Solution:

Possible solutions include $\sinh^{-1}\left(\frac{x}{2}\right) + C$ and $\ln \left| \sqrt{x^2 + 4} + x \right| + C$.

Hint

Answers may vary.

Key Concepts

- An integration table may be used to evaluate indefinite integrals.
- A CAS (or computer algebra system) may be used to evaluate indefinite integrals.
- It may require some effort to reconcile equivalent solutions obtained using different methods.

Use a table of integrals to evaluate the following integrals.

Exercise:

Problem: $\int_0^4 \frac{x}{\sqrt{1 + 2x}} dx$

Exercise:

Problem: $\int \frac{x + 3}{x^2 + 2x + 2} dx$

Solution:

$\frac{1}{2} \ln |x^2 + 2x + 2| + 2 \arctan(x + 1) + C$

Exercise:

Problem: $\int x^3 \sqrt{1 + 2x^2} dx$

Exercise:

Problem: $\int \frac{1}{\sqrt{x^2 + 6x}} dx$

Solution:

$$\cosh^{-1}\left(\frac{x+3}{3}\right) + C$$

Exercise:

Problem: $\int \frac{x}{x+1} dx$

Exercise:

Problem: $\int x \cdot 2^{x^2} dx$

Solution:

$$\frac{2^{x^2-1}}{\ln 2} + C$$

Exercise:

Problem: $\int \frac{1}{4x^2 + 25} dx$

Exercise:

Problem: $\int \frac{dy}{\sqrt{4-y^2}}$

Solution:

$$\arcsin\left(\frac{y}{2}\right) + C$$

Exercise:

Problem: $\int \sin^3(2x) \cos(2x) dx$

Exercise:

Problem: $\int \csc(2w) \cot(2w) dw$

Solution:

$$-\frac{1}{2} \csc(2w) + C$$

Exercise:

Problem: $\int 2^y dy$

Exercise:

Problem: $\int_0^1 \frac{3x dx}{\sqrt{x^2 + 8}}$

Solution:

$$9 - 6\sqrt{2}$$

Exercise:

Problem: $\int_{-1/4}^{1/4} \sec^2(\pi x) \tan(\pi x) dx$

Exercise:

Problem: $\int_0^{\pi/2} \tan^2\left(\frac{x}{2}\right) dx$

Solution:

$$2 - \frac{\pi}{2}$$

Exercise:

Problem: $\int \cos^3 x dx$

Exercise:

Problem: $\int \tan^5(3x) dx$

Solution:

$$\frac{1}{12} \tan^4(3x) - \frac{1}{6} \tan^2(3x) + \frac{1}{3} \ln |\sec(3x)| + C$$

Exercise:

Problem: $\int \sin^2 y \cos^3 y dy$

Use a CAS to evaluate the following integrals. Tables can also be used to verify the answers.

Exercise:

Problem: [T] $\int \frac{dw}{1 + \sec\left(\frac{w}{2}\right)}$

Solution:

$$2 \cot\left(\frac{w}{2}\right) - 2 \csc\left(\frac{w}{2}\right) + w + C$$

Exercise:

Problem: [T] $\int \frac{dw}{1 - \cos(7w)}$

Exercise:

Problem: [T] $\int_0^t \frac{dt}{4 \cos t + 3 \sin t}$

Solution:

$$\frac{1}{5} \ln \left| \frac{2(5+4\sin t-3\cos t)}{4\cos t+3\sin t} \right|$$

Exercise:

Problem: [T] $\int \frac{\sqrt{x^2-9}}{3x} dx$

Exercise:

Problem: [T] $\int \frac{dx}{x^{1/2} + x^{1/3}}$

Solution:

$$6x^{1/6} - 3x^{1/3} + 2\sqrt{x} - 6 \ln [1 + x^{1/6}] + C$$

Exercise:

Problem: [T] $\int \frac{dx}{x\sqrt{x-1}}$

Exercise:

Problem: [T] $\int x^3 \sin x \, dx$

Solution:

$$-x^3 \cos x + 3x^2 \sin x + 6x \cos x - 6 \sin x + C$$

Exercise:

Problem: [T] $\int x \sqrt{x^4 - 9} dx$

Exercise:

Problem: [T] $\int \frac{x}{1 + e^{-x^2}} dx$

Solution:

$$\frac{1}{2} \left(x^2 + \ln |1 + e^{-x^2}| \right) + C$$

Exercise:

Problem: [T] $\int \frac{\sqrt{3 - 5x}}{2x} dx$

Exercise:

Problem: [T] $\int \frac{dx}{x \sqrt{x - 1}}$

Solution:

$$2 \arctan \left(\sqrt{x - 1} \right) + C$$

Exercise:

Problem: [T] $\int e^x \cos^{-1}(e^x) dx$

Use a calculator or CAS to evaluate the following integrals.

Exercise:

Problem: [T] $\int_0^{\pi/4} \cos(2x) dx$

Solution:

$$0.5 = \frac{1}{2}$$

Exercise:

Problem: [T] $\int_0^1 x \cdot e^{-x^2} dx$

Exercise:

Problem: [T] $\int_0^8 \frac{2x}{\sqrt{x^2 + 36}} dx$

Solution:

8.0

Exercise:

Problem: [T] $\int_0^{2/\sqrt{3}} \frac{1}{4 + 9x^2} dx$

Exercise:

Problem: [T] $\int \frac{dx}{x^2 + 4x + 13}$

Solution:

$\frac{1}{3} \arctan\left(\frac{1}{3}(x + 2)\right) + C$

Exercise:

Problem: [T] $\int \frac{dx}{1 + \sin x}$

Use tables to evaluate the integrals. You may need to complete the square or change variables to put the integral into a form given in the table.

Exercise:

Problem: $\int \frac{dx}{x^2 + 2x + 10}$

Solution:

$\frac{1}{3} \arctan\left(\frac{x+1}{3}\right) + C$

Exercise:

Problem: $\int \frac{dx}{\sqrt{x^2 - 6x}}$

Exercise:

Problem: $\int \frac{e^x}{\sqrt{e^{2x} - 4}} dx$

Solution:

$$\ln(e^x + \sqrt{4 + e^{2x}}) + C$$

Exercise:

Problem: $\int \frac{\cos x}{\sin^2 x + 2 \sin x} dx$

Exercise:

Problem: $\int \frac{\arctan(x^3)}{x^4} dx$

Solution:

$$\ln x - \frac{1}{6} \ln(x^6 + 1) - \frac{\arctan(x^3)}{3x^3} + C$$

Exercise:

Problem: $\int \frac{\ln|x| \arcsin(\ln|x|)}{x} dx$

Use tables to perform the integration.

Exercise:

Problem: $\int \frac{dx}{\sqrt{x^2 + 16}}$

Solution:

$$\ln|x + \sqrt{16 + x^2}| + C$$

Exercise:

Problem: $\int \frac{3x}{2x + 7} dx$

Exercise:

Problem: $\int \frac{dx}{1 - \cos(4x)}$

Solution:

$$-\frac{1}{4} \cot(2x) + C$$

Exercise:

Problem: $\int \frac{dx}{\sqrt{4x+1}}$

Exercise:**Problem:**

Find the area bounded by $y(4 + 25x^2) = 5$, $x = 0$, $y = 0$, and $x = 4$. Use a table of integrals or a CAS.

Solution:

$$\frac{1}{2} \arctan 10$$

Exercise:**Problem:**

The region bounded between the curve $y = \frac{1}{\sqrt{1+\cos x}}$, $0.3 \leq x \leq 1.1$, and the x -axis is revolved about the x -axis to generate a solid. Use a table of integrals to find the volume of the solid generated. (Round the answer to two decimal places.)

Exercise:**Problem:**

Use substitution and a table of integrals to find the area of the surface generated by revolving the curve $y = e^x$, $0 \leq x \leq 3$, about the x -axis. (Round the answer to two decimal places.)

Solution:

$$1276.14$$

Exercise:**Problem:**

[T] Use an integral table and a calculator to find the area of the surface generated by revolving the curve $y = \frac{x^2}{2}$, $0 \leq x \leq 1$, about the x -axis. (Round the answer to two decimal places.)

Exercise:

Problem:

[T] Use a CAS or tables to find the area of the surface generated by revolving the curve $y = \cos x$, $0 \leq x \leq \frac{\pi}{2}$, about the x -axis. (Round the answer to two decimal places.)

Solution:

7.21

Exercise:

Problem: Find the length of the curve $y = \frac{x^2}{4}$ over $[0, 8]$.

Exercise:

Problem: Find the length of the curve $y = e^x$ over $[0, \ln(2)]$.

Solution:

$$\sqrt{5} - \sqrt{2} + \ln \left| \frac{2+2\sqrt{2}}{1+\sqrt{5}} \right|$$

Exercise:**Problem:**

Find the area of the surface formed by revolving the graph of $y = 2\sqrt{x}$ over the interval $[0, 9]$ about the x -axis.

Exercise:

Problem: Find the average value of the function $f(x) = \frac{1}{x^2+1}$ over the interval $[-3, 3]$.

Solution:

$$\frac{1}{3} \arctan(3) \approx 0.416$$

Exercise:**Problem:**

Approximate the arc length of the curve $y = \tan(\pi x)$ over the interval $[0, \frac{1}{4}]$. (Round the answer to three decimal places.)

Glossary

computer algebra system (CAS)

technology used to perform many mathematical tasks, including integration

integration table

a table that lists integration formulas

Numerical Integration

- Approximate the value of a definite integral by using the midpoint and trapezoidal rules.
- Determine the absolute and relative error in using a numerical integration technique.
- Estimate the absolute and relative error using an error-bound formula.
- Recognize when the midpoint and trapezoidal rules over- or underestimate the true value of an integral.
- Use Simpson's rule to approximate the value of a definite integral to a given accuracy.

The antiderivatives of many functions either cannot be expressed or cannot be expressed easily in closed form (that is, in terms of known functions). Consequently, rather than evaluate definite integrals of these functions directly, we resort to various techniques of **numerical integration** to approximate their values. In this section we explore several of these techniques. In addition, we examine the process of estimating the error in using these techniques.

The Midpoint Rule

Earlier in this text we defined the definite integral of a function over an interval as the limit of Riemann sums. In general, any Riemann sum of a function $f(x)$ over an interval $[a, b]$ may be viewed as an estimate of $\int_a^b f(x)dx$.

Recall that a Riemann sum of a function $f(x)$ over an interval $[a, b]$ is obtained by selecting a partition

Equation:

$$P = \{x_0, x_1, x_2, \dots, x_n\}, \text{ where } a = x_0 < x_1 < x_2 < \dots < x_n = b$$

and a set

Equation:

$$S = \{x_1^*, x_2^*, \dots, x_n^*\}, \text{ where } x_{i-1} \leq x_i^* \leq x_i \text{ for all } i.$$

The Riemann sum corresponding to the partition P and the set S is given by $\sum_{i=1}^n f(x_i^*)\Delta x_i$, where

$\Delta x_i = x_i - x_{i-1}$, the length of the i th subinterval.

The **midpoint rule** for estimating a definite integral uses a Riemann sum with subintervals of equal width and the midpoints, m_i , of each subinterval in place of x_i^* . Formally, we state a theorem regarding the convergence of the midpoint rule as follows.

Note:

The Midpoint Rule

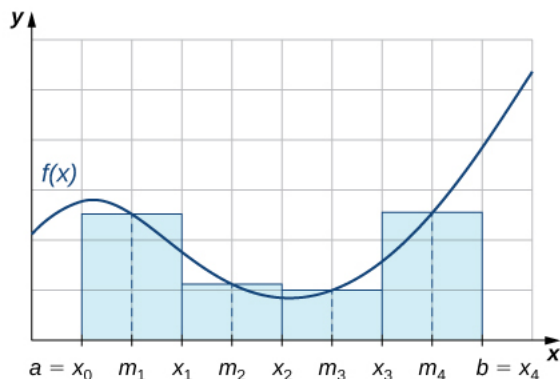
Assume that $f(x)$ is continuous on $[a, b]$. Let n be a positive integer and $\Delta x = \frac{b-a}{n}$. If $[a, b]$ is divided into n subintervals, each of length Δx , and m_i is the midpoint of the i th subinterval, set

Equation:

$$M_n = \sum_{i=1}^n f(m_i)\Delta x.$$

Then $\lim_{n \rightarrow \infty} M_n = \int_a^b f(x)dx$.

As we can see in [\[link\]](#), if $f(x) \geq 0$ over $[a, b]$, then $\sum_{i=1}^n f(m_i)\Delta x$ corresponds to the sum of the areas of rectangles approximating the area between the graph of $f(x)$ and the x -axis over $[a, b]$. The graph shows the rectangles corresponding to M_4 for a nonnegative function over a closed interval $[a, b]$.



The midpoint rule approximates the area between the graph of $f(x)$ and the x -axis by summing the areas of rectangles with midpoints that are points on $f(x)$.

Example:

Exercise:

Problem:

Using the Midpoint Rule with M_4

Use the midpoint rule to estimate $\int_0^1 x^2 dx$ using four subintervals. Compare the result with the actual value of this integral.

Solution:

Each subinterval has length $\Delta x = \frac{1-0}{4} = \frac{1}{4}$. Therefore, the subintervals consist of

Equation:

$$\left[0, \frac{1}{4}\right], \left[\frac{1}{4}, \frac{1}{2}\right], \left[\frac{1}{2}, \frac{3}{4}\right], \text{ and } \left[\frac{3}{4}, 1\right].$$

The midpoints of these subintervals are $\left\{\frac{1}{8}, \frac{3}{8}, \frac{5}{8}, \frac{7}{8}\right\}$. Thus,

Equation:

$$M_4 = \frac{1}{4}f\left(\frac{1}{8}\right) + \frac{1}{4}f\left(\frac{3}{8}\right) + \frac{1}{4}f\left(\frac{5}{8}\right) + \frac{1}{4}f\left(\frac{7}{8}\right) = \frac{1}{4} \cdot \frac{1}{64} + \frac{1}{4} \cdot \frac{9}{64} + \frac{1}{4} \cdot \frac{25}{64} + \frac{1}{4} \cdot \frac{21}{64} = \frac{21}{64}.$$

Since

Equation:

$$\int_0^1 x^2 dx = \frac{1}{3} \text{ and } \left| \frac{1}{3} - \frac{21}{64} \right| = \frac{1}{192} \approx 0.0052,$$

we see that the midpoint rule produces an estimate that is somewhat close to the actual value of the definite integral.

Example:

Exercise:

Problem:

Using the Midpoint Rule with M_6

Use M_6 to estimate the length of the curve $y = \frac{1}{2}x^2$ on $[1, 4]$.

Solution:

The length of $y = \frac{1}{2}x^2$ on $[1, 4]$ is

Equation:

$$\int_1^4 \sqrt{1 + \left(\frac{dy}{dx}\right)^2} dx.$$

Since $\frac{dy}{dx} = x$, this integral becomes $\int_1^4 \sqrt{1 + x^2} dx$.

If $[1, 4]$ is divided into six subintervals, then each subinterval has length $\Delta x = \frac{4-1}{6} = \frac{1}{2}$ and the midpoints of the subintervals are $\left\{ \frac{5}{4}, \frac{7}{4}, \frac{9}{4}, \frac{11}{4}, \frac{13}{4}, \frac{15}{4} \right\}$. If we set $f(x) = \sqrt{1 + x^2}$,

Equation:

$$\begin{aligned} M_6 &= \frac{1}{2}f\left(\frac{5}{4}\right) + \frac{1}{2}f\left(\frac{7}{4}\right) + \frac{1}{2}f\left(\frac{9}{4}\right) + \frac{1}{2}f\left(\frac{11}{4}\right) + \frac{1}{2}f\left(\frac{13}{4}\right) + \frac{1}{2}f\left(\frac{15}{4}\right) \\ &\approx \frac{1}{2}(1.6008 + 2.0156 + 2.4622 + 2.9262 + 3.4004 + 3.8810) = 8.1431. \end{aligned}$$

Note:

Exercise:

Problem: Use the midpoint rule with $n = 2$ to estimate $\int_1^2 \frac{1}{x} dx$.

Solution:

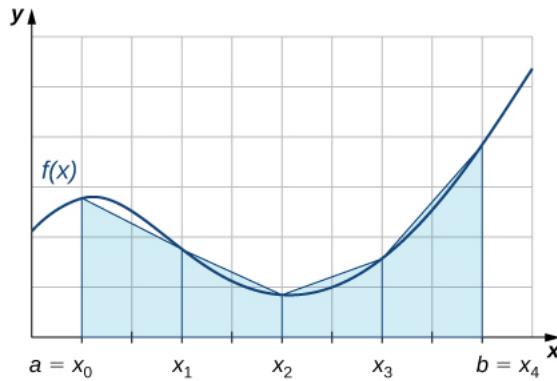
$$\frac{24}{35}$$

Hint

$$\Delta x = \frac{1}{2}, m_1 = \frac{5}{4}, \text{ and } m_2 = \frac{7}{4}.$$

The Trapezoidal Rule

We can also approximate the value of a definite integral by using trapezoids rather than rectangles. In [\[link\]](#), the area beneath the curve is approximated by trapezoids rather than by rectangles.



Trapezoids may be used to approximate the area under a curve, hence approximating the definite integral.

The **trapezoidal rule** for estimating definite integrals uses trapezoids rather than rectangles to approximate the area under a curve. To gain insight into the final form of the rule, consider the trapezoids shown in [\[link\]](#). We assume that the length of each subinterval is given by Δx . First, recall that the area of a trapezoid with a height of h and bases of length b_1 and b_2 is given by $\text{Area} = \frac{1}{2}h(b_1 + b_2)$. We see that the first trapezoid has a height Δx and parallel bases of length $f(x_0)$ and $f(x_1)$. Thus, the area of the first trapezoid in [\[link\]](#) is

Equation:

$$\frac{1}{2}\Delta x(f(x_0) + f(x_1)).$$

The areas of the remaining three trapezoids are

Equation:

$$\frac{1}{2}\Delta x(f(x_1) + f(x_2)), \frac{1}{2}\Delta x(f(x_2) + f(x_3)), \text{ and } \frac{1}{2}\Delta x(f(x_3) + f(x_4)).$$

Consequently,

Equation:

$$\int_a^b f(x)dx \approx \frac{1}{2}\Delta x(f(x_0) + f(x_1)) + \frac{1}{2}\Delta x(f(x_1) + f(x_2)) + \frac{1}{2}\Delta x(f(x_2) + f(x_3)) + \frac{1}{2}\Delta x(f(x_3) + f(x_4)).$$

After taking out a common factor of $\frac{1}{2}\Delta x$ and combining like terms, we have

Equation:

$$\int_a^b f(x)dx \approx \frac{1}{2}\Delta x (f(x_0) + 2f(x_1) + 2f(x_2) + 2f(x_3) + f(x_4)).$$

Generalizing, we formally state the following rule.

Note:

The Trapezoidal Rule

Assume that $f(x)$ is continuous over $[a, b]$. Let n be a positive integer and $\Delta x = \frac{b-a}{n}$. Let $[a, b]$ be divided into n subintervals, each of length Δx , with endpoints at $P = \{x_0, x_1, x_2, \dots, x_n\}$. Set

Equation:

$$T_n = \frac{1}{2} \Delta x (f(x_0) + 2f(x_1) + 2f(x_2) + \dots + 2f(x_{n-1}) + f(x_n)).$$

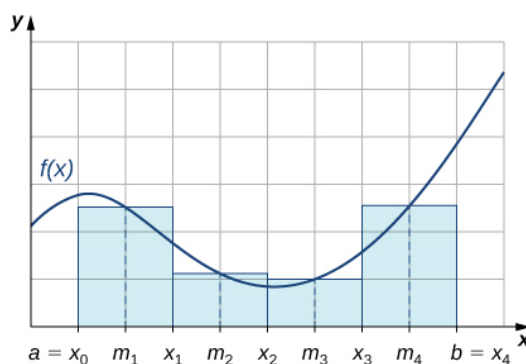
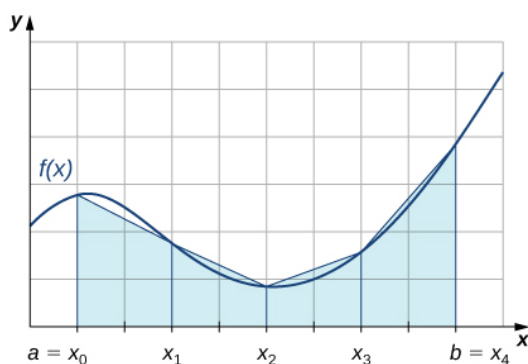
Then, $\lim_{n \rightarrow +\infty} T_n = \int_a^b f(x) dx$.

Before continuing, let's make a few observations about the trapezoidal rule. First of all, it is useful to note that

Equation:

$$T_n = \frac{1}{2} (L_n + R_n) \text{ where } L_n = \sum_{i=1}^n f(x_{i-1}) \Delta x \text{ and } R_n = \sum_{i=1}^n f(x_i) \Delta x.$$

That is, L_n and R_n approximate the integral using the left-hand and right-hand endpoints of each subinterval, respectively. In addition, a careful examination of [\[link\]](#) leads us to make the following observations about using the trapezoidal rules and midpoint rules to estimate the definite integral of a nonnegative function. The trapezoidal rule tends to overestimate the value of a definite integral systematically over intervals where the function is concave up and to underestimate the value of a definite integral systematically over intervals where the function is concave down. On the other hand, the midpoint rule tends to average out these errors somewhat by partially overestimating and partially underestimating the value of the definite integral over these same types of intervals. This leads us to hypothesize that, in general, the midpoint rule tends to be more accurate than the trapezoidal rule.



The trapezoidal rule tends to be less accurate than the midpoint rule.

Example:

Exercise:**Problem:**
Using the Trapezoidal Rule

Use the trapezoidal rule to estimate $\int_0^1 x^2 dx$ using four subintervals.

Solution:

The endpoints of the subintervals consist of elements of the set $P = \{0, \frac{1}{4}, \frac{1}{2}, \frac{3}{4}, 1\}$ and $\Delta x = \frac{1-0}{4} = \frac{1}{4}$. Thus,

Equation:

$$\begin{aligned}\int_0^1 x^2 dx &\approx \frac{1}{2} \cdot \frac{1}{4} \left(f(0) + 2f\left(\frac{1}{4}\right) + 2f\left(\frac{1}{2}\right) + 2f\left(\frac{3}{4}\right) + f(1) \right) \\ &= \frac{1}{8} \left(0 + 2 \cdot \frac{1}{16} + 2 \cdot \frac{1}{4} + 2 \cdot \frac{9}{16} + 1 \right) \\ &= \frac{11}{32}.\end{aligned}$$

Note:**Exercise:**

Problem: Use the trapezoidal rule with $n = 2$ to estimate $\int_1^2 \frac{1}{x} dx$.

Solution:

$$\frac{17}{24}$$

Hint

Set $\Delta x = \frac{1}{2}$. The endpoints of the subintervals are the elements of the set $P = \{1, \frac{3}{2}, 2\}$.

Absolute and Relative Error

An important aspect of using these numerical approximation rules consists of calculating the error in using them for estimating the value of a definite integral. We first need to define **absolute error** and **relative error**.

Note:**Definition**

If B is our estimate of some quantity having an actual value of A , then the absolute error is given by $|A - B|$. The relative error is the error as a percentage of the absolute value and is given by $\left| \frac{A-B}{A} \right| = \left| \frac{A-B}{A} \right| \cdot 100\%$.

Example:**Exercise:****Problem:**

Calculating Error in the Midpoint Rule

Calculate the absolute and relative error in the estimate of $\int_0^1 x^2 dx$ using the midpoint rule, found in [\[link\]](#).

Solution:

The calculated value is $\int_0^1 x^2 dx = \frac{1}{3}$ and our estimate from the example is $M_4 = \frac{21}{64}$. Thus, the absolute error is given by $\left| \left(\frac{1}{3} \right) - \left(\frac{21}{64} \right) \right| = \frac{1}{192} \approx 0.0052$. The relative error is

Equation:

$$\frac{1/192}{1/3} = \frac{1}{64} \approx 0.015625 \approx 1.6\%.$$

Example:

Exercise:

Problem:

Calculating Error in the Trapezoidal Rule

Calculate the absolute and relative error in the estimate of $\int_0^1 x^2 dx$ using the trapezoidal rule, found in [\[link\]](#).

Solution:

The calculated value is $\int_0^1 x^2 dx = \frac{1}{3}$ and our estimate from the example is $T_4 = \frac{11}{32}$. Thus, the absolute error is given by $\left| \frac{1}{3} - \frac{11}{32} \right| = \frac{1}{96} \approx 0.0104$. The relative error is given by

Equation:

$$\frac{1/96}{1/3} = 0.03125 \approx 3.1\%.$$

Note:

Exercise:

Problem:

In an earlier checkpoint, we estimated $\int_1^2 \frac{1}{x} dx$ to be $\frac{24}{35}$ using T_2 . The actual value of this integral is $\ln 2$. Using $\frac{24}{35} \approx 0.6857$ and $\ln 2 \approx 0.6931$, calculate the absolute error and the relative error.

Solution:

0.0074, 1.1%

Hint

Use the previous examples as a guide.

In the two previous examples, we were able to compare our estimate of an integral with the actual value of the integral; however, we do not typically have this luxury. In general, if we are approximating an integral, we are doing so because we cannot compute the exact value of the integral itself easily. Therefore, it is often helpful to be able to determine an upper bound for the error in an approximation of an integral. The following theorem provides error bounds for the midpoint and trapezoidal rules. The theorem is stated without proof.

Note:

Error Bounds for the Midpoint and Trapezoidal Rules

Let $f(x)$ be a continuous function over $[a, b]$, having a second derivative $f''(x)$ over this interval. If M is the maximum value of $|f''(x)|$ over $[a, b]$, then the upper bounds for the error in using M_n and T_n to estimate

$\int_a^b f(x) dx$ are

Equation:

$$\text{Error in } M_n \leq \frac{M(b-a)^3}{24n^2}$$

and

Equation:

$$\text{Error in } T_n \leq \frac{M(b-a)^3}{12n^2}.$$

We can use these bounds to determine the value of n necessary to guarantee that the error in an estimate is less than a specified value.

Example:

Exercise:

Problem:

Determining the Number of Intervals to Use

What value of n should be used to guarantee that an estimate of $\int_0^1 e^{x^2} dx$ is accurate to within 0.01 if we use the midpoint rule?

Solution:

We begin by determining the value of M , the maximum value of $|f''(x)|$ over $[0, 1]$ for $f(x) = e^{x^2}$. Since $f'(x) = 2xe^{x^2}$, we have

Equation:

$$f''(x) = 2e^{x^2} + 4x^2e^{x^2}.$$

Thus,

Equation:

$$|f''(x)| = 2e^{x^2} (1 + 2x^2) \leq 2 \cdot e \cdot 3 = 6e.$$

From the error-bound [\[link\]](#), we have

Equation:

$$\text{Error in } M_n \leq \frac{M(b-a)^3}{24n^2} \leq \frac{6e(1-0)^3}{24n^2} = \frac{6e}{24n^2}.$$

Now we solve the following inequality for n :

Equation:

$$\frac{6e}{24n^2} \leq 0.01.$$

Thus, $n \geq \sqrt{\frac{600e}{24}} \approx 8.24$. Since n must be an integer satisfying this inequality, a choice of $n = 9$ would guarantee that $\left| \int_0^1 e^{x^2} dx - M_n \right| < 0.01$.

Analysis

We might have been tempted to round 8.24 down and choose $n = 8$, but this would be incorrect because we must have an integer greater than or equal to 8.24. We need to keep in mind that the error estimates provide an upper bound only for the error. The actual estimate may, in fact, be a much better approximation than is indicated by the error bound.

Note:**Exercise:**

Problem: Use [\[link\]](#) to find an upper bound for the error in using M_4 to estimate $\int_0^1 x^2 dx$.

Solution:

$$\frac{1}{192}$$

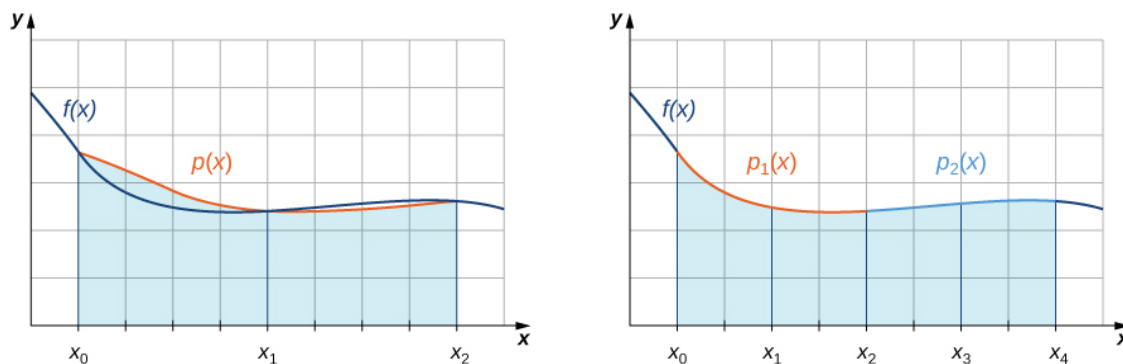
Hint

$f''(x) = 2$, so $M = 2$.

Simpson's Rule

With the midpoint rule, we estimated areas of regions under curves by using rectangles. In a sense, we approximated the curve with piecewise constant functions. With the trapezoidal rule, we approximated the curve by using piecewise linear functions. What if we were, instead, to approximate a curve using piecewise quadratic functions? With **Simpson's rule**, we do just this. We partition the interval into an even number of subintervals, each of equal width. Over the first pair of subintervals we approximate $\int_{x_0}^{x_2} f(x) dx$ with $\int_{x_0}^{x_2} p(x) dx$, where

$p(x) = Ax^2 + Bx + C$ is the quadratic function passing through $(x_0, f(x_0))$, $(x_1, f(x_1))$, and $(x_2, f(x_2))$ ([link](#)). Over the next pair of subintervals we approximate $\int_{x_2}^{x_4} f(x)dx$ with the integral of another quadratic function passing through $(x_2, f(x_2))$, $(x_3, f(x_3))$, and $(x_4, f(x_4))$. This process is continued with each successive pair of subintervals.



With Simpson's rule, we approximate a definite integral by integrating a piecewise quadratic function.

To understand the formula that we obtain for Simpson's rule, we begin by deriving a formula for this approximation over the first two subintervals. As we go through the derivation, we need to keep in mind the following relationships:

Equation:

$$f(x_0) = p(x_0) = Ax_0^2 + Bx_0 + C$$

$$f(x_1) = p(x_1) = Ax_1^2 + Bx_1 + C$$

$$f(x_2) = p(x_2) = Ax_2^2 + Bx_2 + C$$

$x_2 - x_0 = 2\Delta x$, where Δx is the length of a subinterval.

Equation:

$$x_2 + x_0 = 2x_1, \text{ since } x_1 = \frac{(x_2 + x_0)}{2}.$$

Thus,

Equation:

$$\begin{aligned}
\int_{x_0}^{x_2} f(x) dx &\approx \int_{x_0}^{x_2} p(x) dx \\
&= \int_{x_0}^{x_2} (Ax^2 + Bx + C) dx \\
&= \left. \frac{A}{3} x^3 + \frac{B}{2} x^2 + Cx \right|_{x_0}^{x_2} \\
&= \frac{A}{3} (x_2^3 - x_0^3) + \frac{B}{2} (x_2^2 - x_0^2) + C(x_2 - x_0) \\
&= \frac{A}{3} (x_2 - x_0) (x_2^2 + x_2 x_0 + x_0^2) \\
&\quad + \frac{B}{2} (x_2 - x_0) (x_2 + x_0) + C(x_2 - x_0) \\
&= \frac{x_2 - x_0}{6} (2A(x_2^2 + x_2 x_0 + x_0^2) + 3B(x_2 + x_0) + 6C) \\
&= \frac{\Delta x}{3} ((Ax_2^2 + Bx_2 + C) + (Ax_0^2 + Bx_0 + C) \\
&\quad + A(x_2^2 + 2x_2 x_0 + x_0^2) + 2B(x_2 + x_0) + 4C) \\
&= \frac{\Delta x}{3} (f(x_2) + f(x_0) + A(x_2 + x_0)^2 + 2B(x_2 + x_0) + 4C) \\
&= \frac{\Delta x}{3} (f(x_2) + f(x_0) + A(2x_1)^2 + 2B(2x_1) + 4C) \\
&= \frac{\Delta x}{3} (f(x_2) + f(x_0) + 4f(x_1) + f(x_0)).
\end{aligned}$$

Find the antiderivative.

Evaluate the antiderivative.

Factor out $\frac{x_2 - x_0}{6}$.

Rearrange the terms.

Factor and substitute.

$$f(x_2) = Ax_2^2 + Bx_2 + C$$

$$f(x_0) = Ax_0^2 + Bx_0 + C$$

Substitute $x_2 + x_0 = 2x_1$.

Expand and substitute

$$f(x_1) = Ax_1^2 + Bx_1 + C.$$

If we approximate $\int_{x_2}^{x_4} f(x) dx$ using the same method, we see that we have

Equation:

$$\int_{x_0}^{x_4} f(x) dx \approx \frac{\Delta x}{3} (f(x_4) + 4f(x_3) + f(x_2)).$$

Combining these two approximations, we get

Equation:

$$\int_{x_0}^{x_4} f(x) dx = \frac{\Delta x}{3} (f(x_0) + 4f(x_1) + 2f(x_2) + 4f(x_3) + f(x_4)).$$

The pattern continues as we add pairs of subintervals to our approximation. The general rule may be stated as follows.

Note:

Simpson's Rule

Assume that $f(x)$ is continuous over $[a, b]$. Let n be a positive even integer and $\Delta x = \frac{b-a}{n}$. Let $[a, b]$ be divided into n subintervals, each of length Δx , with endpoints at $P = \{x_0, x_1, x_2, \dots, x_n\}$. Set

Equation:

$$S_n = \frac{\Delta x}{3} (f(x_0) + 4f(x_1) + 2f(x_2) + 4f(x_3) + 2f(x_4) + \dots + 2f(x_{n-2}) + 4f(x_{n-1}) + f(x_n)).$$

Then,

Equation:

$$\lim_{n \rightarrow +\infty} S_n = \int_a^b f(x) dx.$$

Just as the trapezoidal rule is the average of the left-hand and right-hand rules for estimating definite integrals, Simpson's rule may be obtained from the midpoint and trapezoidal rules by using a weighted average. It can be shown that $S_{2n} = \left(\frac{2}{3}\right)M_n + \left(\frac{1}{3}\right)T_n$.

It is also possible to put a bound on the error when using Simpson's rule to approximate a definite integral. The bound in the error is given by the following rule:

Note:

Rule: Error Bound for Simpson's Rule

Let $f(x)$ be a continuous function over $[a, b]$ having a fourth derivative, $f^{(4)}(x)$, over this interval. If M is the maximum value of $|f^{(4)}(x)|$ over $[a, b]$, then the upper bound for the error in using S_n to estimate $\int_a^b f(x) dx$ is given by

Equation:

$$\text{Error in } S_n \leq \frac{M(b-a)^5}{180n^4}.$$

Example:

Exercise:

Problem:

Applying Simpson's Rule 1

Use S_2 to approximate $\int_0^1 x^3 dx$. Estimate a bound for the error in S_2 .

Solution:

Since $[0, 1]$ is divided into two intervals, each subinterval has length $\Delta x = \frac{1-0}{2} = \frac{1}{2}$. The endpoints of these subintervals are $\{0, \frac{1}{2}, 1\}$. If we set $f(x) = x^3$, then

$S_4 = \frac{1}{3} \cdot \frac{1}{2} (f(0) + 4f(\frac{1}{2}) + f(1)) = \frac{1}{6} (0 + 4 \cdot \frac{1}{8} + 1) = \frac{1}{4}$. Since $f^{(4)}(x) = 0$ and consequently $M = 0$, we see that

Equation:

$$\text{Error in } S_2 \leq \frac{0(1)^5}{180 \cdot 2^4} = 0.$$

This bound indicates that the value obtained through Simpson's rule is exact. A quick check will verify that, in fact, $\int_0^1 x^3 dx = \frac{1}{4}$.

Example:

Exercise:

Problem:

Applying Simpson's Rule 2

Use S_6 to estimate the length of the curve $y = \frac{1}{2}x^2$ over $[1, 4]$.

Solution:

The length of $y = \frac{1}{2}x^2$ over $[1, 4]$ is $\int_1^4 \sqrt{1+x^2} dx$. If we divide $[1, 4]$ into six subintervals, then each subinterval has length $\Delta x = \frac{4-1}{6} = \frac{1}{2}$, and the endpoints of the subintervals are $\{1, \frac{3}{2}, 2, \frac{5}{2}, 3, \frac{7}{2}, 4\}$. Setting $f(x) = \sqrt{1+x^2}$,

Equation:

$$S_6 = \frac{1}{3} \cdot \frac{1}{2} \left(f(1) + 4f\left(\frac{3}{2}\right) + 2f(2) + 4f\left(\frac{5}{2}\right) + 2f(3) + 4f\left(\frac{7}{2}\right) + f(4) \right).$$

After substituting, we have

Equation:

$$S_6 = \frac{1}{6} (1.4142 + 4 \cdot 1.80278 + 2 \cdot 2.23607 + 4 \cdot 2.69258 + 2 \cdot 3.16228 + 4 \cdot 3.64005 + 4.12311) \\ \approx 8.14594.$$

Note:

Exercise:

Problem: Use S_2 to estimate $\int_1^2 \frac{1}{x} dx$.

Solution:

$$\frac{25}{36}$$

Hint

$$S_2 = \left(\frac{1}{3} \Delta x (f(x_0) + 4f(x_1) + f(x_2)) \right)$$

Key Concepts

- We can use numerical integration to estimate the values of definite integrals when a closed form of the integral is difficult to find or when an approximate value only of the definite integral is needed.

- The most commonly used techniques for numerical integration are the midpoint rule, trapezoidal rule, and Simpson's rule.
- The midpoint rule approximates the definite integral using rectangular regions whereas the trapezoidal rule approximates the definite integral using trapezoidal approximations.
- Simpson's rule approximates the definite integral by first approximating the original function using piecewise quadratic functions.

Key Equations

- **Midpoint rule**

$$M_n = \sum_{i=1}^n f(m_i) \Delta x$$

- **Trapezoidal rule**

$$T_n = \frac{1}{2} \Delta x (f(x_0) + 2f(x_1) + 2f(x_2) + \cdots + 2f(x_{n-1}) + f(x_n))$$

- **Simpson's rule**

$$S_n = \frac{\Delta x}{3} (f(x_0) + 4f(x_1) + 2f(x_2) + 4f(x_3) + 2f(x_4) + 4f(x_5) + \cdots + 2f(x_{n-2}) + 4f(x_{n-1}) + f(x_n))$$

- **Error bound for midpoint rule**

$$\text{Error in } M_n \leq \frac{M(b-a)^3}{24n^2}$$

- **Error bound for trapezoidal rule**

$$\text{Error in } T_n \leq \frac{M(b-a)^3}{12n^2}$$

- **Error bound for Simpson's rule**

$$\text{Error in } S_n \leq \frac{M(b-a)^5}{180n^4}$$

Approximate the following integrals using either the midpoint rule, trapezoidal rule, or Simpson's rule as indicated. (Round answers to three decimal places.)

Exercise:

Problem: $\int_1^2 \frac{dx}{x}$; trapezoidal rule; $n = 5$

Solution:

0.696

Exercise:

Problem: $\int_0^3 \sqrt{4+x^3} dx$; trapezoidal rule; $n = 6$

Exercise:

Problem: $\int_0^3 \sqrt{4+x^3} dx$; trapezoidal rule; $n = 3$

Solution:

9.298

Exercise:

Problem: $\int_0^{12} x^2 dx$; midpoint rule; $n = 6$

Exercise:

Problem: $\int_0^1 \sin^2(\pi x) dx$; midpoint rule; $n = 3$

Solution:

0.5000

Exercise:

Problem: Use the midpoint rule with eight subdivisions to estimate $\int_2^4 x^2 dx$.

Exercise:

Problem: Use the trapezoidal rule with four subdivisions to estimate $\int_2^4 x^2 dx$.

Solution:

$T_4 = 18.75$

Exercise:

Problem:

Find the exact value of $\int_2^4 x^2 dx$. Find the error of approximation between the exact value and the value calculated using the trapezoidal rule with four subdivisions. Draw a graph to illustrate.

Approximate the integral to three decimal places using the indicated rule.

Exercise:

Problem: $\int_0^1 \sin^2(\pi x) dx$; trapezoidal rule; $n = 6$

Solution:

0.500

Exercise:

Problem: $\int_0^3 \frac{1}{1+x^3} dx$; trapezoidal rule; $n = 6$

Exercise:

Problem: $\int_0^3 \frac{1}{1+x^3} dx$; trapezoidal rule; $n = 3$

Solution:

1.2819

Exercise:

Problem: $\int_0^{0.8} e^{-x^2} dx$; trapezoidal rule; $n = 4$

Exercise:

Problem: $\int_0^{0.8} e^{-x^2} dx$; Simpson's rule; $n = 4$

Solution:

0.6577

Exercise:

Problem: $\int_0^{0.4} \sin(x^2) dx$; trapezoidal rule; $n = 4$

Exercise:

Problem: $\int_0^{0.4} \sin(x^2) dx$; Simpson's rule; $n = 4$

Solution:

0.0213

Exercise:

Problem: $\int_{0.1}^{0.5} \frac{\cos x}{x} dx$; trapezoidal rule; $n = 4$

Exercise:

Problem: $\int_{0.1}^{0.5} \frac{\cos x}{x} dx$; Simpson's rule; $n = 4$

Solution:

1.5629

Exercise:

Problem:

Evaluate $\int_0^1 \frac{dx}{1+x^2}$ exactly and show that the result is $\pi/4$. Then, find the approximate value of the integral using the trapezoidal rule with $n = 4$ subdivisions. Use the result to approximate the value of π .

Exercise:

Problem: Approximate $\int_2^4 \frac{1}{\ln x} dx$ using the midpoint rule with four subdivisions to four decimal places.

Solution:

1.9133

Exercise:

Problem: Approximate $\int_2^4 \frac{1}{\ln x} dx$ using the trapezoidal rule with eight subdivisions to four decimal places.

Exercise:

Problem: Use the trapezoidal rule with four subdivisions to estimate $\int_0^{0.8} x^3 dx$ to four decimal places.

Solution:

$$T(4) = 0.1088$$

Exercise:

Problem:

Use the trapezoidal rule with four subdivisions to estimate $\int_0^{0.8} x^3 dx$. Compare this value with the exact value and find the error estimate.

Exercise:

Problem: Using Simpson's rule with four subdivisions, find $\int_0^{\pi/2} \cos(x) dx$.

Solution:

$$1.0$$

Exercise:

Problem:

Show that the exact value of $\int_0^1 x e^{-x} dx = 1 - \frac{2}{e}$. Find the absolute error if you approximate the integral using the midpoint rule with 16 subdivisions.

Exercise:

Problem:

Given $\int_0^1 x e^{-x} dx = 1 - \frac{2}{e}$, use the trapezoidal rule with 16 subdivisions to approximate the integral and find the absolute error.

Solution:

Approximate error is 0.000325.

Exercise:

Problem:

Find an upper bound for the error in estimating $\int_0^3 (5x + 4) dx$ using the trapezoidal rule with six steps.

Exercise:**Problem:**

Find an upper bound for the error in estimating $\int_4^5 \frac{1}{(x-1)^2} dx$ using the trapezoidal rule with seven subdivisions.

Solution:

$$\frac{1}{7938}$$

Exercise:**Problem:**

Find an upper bound for the error in estimating $\int_0^3 (6x^2 - 1) dx$ using Simpson's rule with $n = 10$ steps.

Exercise:**Problem:**

Find an upper bound for the error in estimating $\int_2^5 \frac{1}{x-1} dx$ using Simpson's rule with $n = 10$ steps.

Solution:

$$\frac{81}{25,000}$$

Exercise:**Problem:**

Find an upper bound for the error in estimating $\int_0^\pi 2x \cos(x) dx$ using Simpson's rule with four steps.

Exercise:**Problem:**

Estimate the minimum number of subintervals needed to approximate the integral $\int_1^4 (5x^2 + 8) dx$ with an error magnitude of less than 0.0001 using the trapezoidal rule.

Solution:

$$475$$

Exercise:**Problem:**

Determine a value of n such that the trapezoidal rule will approximate $\int_0^1 \sqrt{1+x^2} dx$ with an error of no more than 0.01.

Exercise:

Problem:

Estimate the minimum number of subintervals needed to approximate the integral $\int_2^3 (2x^3 + 4x) dx$ with an error of magnitude less than 0.0001 using the trapezoidal rule.

Solution:

174

Exercise:**Problem:**

Estimate the minimum number of subintervals needed to approximate the integral $\int_3^4 \frac{1}{(x-1)^2} dx$ with an error magnitude of less than 0.0001 using the trapezoidal rule.

Exercise:**Problem:**

Use Simpson's rule with four subdivisions to approximate the area under the probability density function $y = \frac{1}{\sqrt{2\pi}} e^{-x^2/2}$ from $x = 0$ to $x = 0.4$.

Solution:

0.1544

Exercise:**Problem:**

Use Simpson's rule with $n = 14$ to approximate (to three decimal places) the area of the region bounded by the graphs of $y = 0$, $x = 0$, and $x = \pi/2$.

Exercise:**Problem:**

The length of one arch of the curve $y = 3 \sin(2x)$ is given by $L = \int_0^{\pi/2} \sqrt{1 + 36 \cos^2(2x)} dx$. Estimate L using the trapezoidal rule with $n = 6$.

Solution:

6.2807

Exercise:**Problem:**

The length of the ellipse $x = a \cos(t)$, $y = b \sin(t)$, $0 \leq t \leq 2\pi$ is given by

$L = 4a \int_0^{\pi/2} \sqrt{1 - e^2 \cos^2(t)} dt$, where e is the eccentricity of the ellipse. Use Simpson's rule with $n = 6$ subdivisions to estimate the length of the ellipse when $a = 2$ and $e = 1/3$.

Exercise:

Problem:

Estimate the area of the surface generated by revolving the curve $y = \cos(2x)$, $0 \leq x \leq \frac{\pi}{4}$ about the x -axis. Use the trapezoidal rule with six subdivisions.

Solution:

4.606

Exercise:**Problem:**

Estimate the area of the surface generated by revolving the curve $y = 2x^2$, $0 \leq x \leq 3$ about the x -axis. Use Simpson's rule with $n = 6$.

Exercise:**Problem:**

The growth rate of a certain tree (in feet) is given by $y = \frac{2}{t+1} + e^{-t^2/2}$, where t is time in years. Estimate the growth of the tree through the end of the second year by using Simpson's rule, using two subintervals. (Round the answer to the nearest hundredth.)

Solution:

3.41 ft

Exercise:**Problem:**

[T] Use a calculator to approximate $\int_0^1 \sin(\pi x) dx$ using the midpoint rule with 25 subdivisions. Compute the relative error of approximation.

Exercise:**Problem:**

[T] Given $\int_1^5 (3x^2 - 2x) dx = 100$, approximate the value of this integral using the trapezoidal rule with 16 subdivisions and determine the absolute error.

Solution:

$T_{16} = 100.125$; absolute error = 0.125

Exercise:**Problem:**

Given that we know the Fundamental Theorem of Calculus, why would we want to develop numerical methods for definite integrals?

Exercise:**Problem:**

The table represents the coordinates (x, y) that give the boundary of a lot. The units of measurement are meters. Use the trapezoidal rule to estimate the number of square meters of land that is in this lot.

x	y	x	y
0	125	600	95
100	125	700	88
200	120	800	75
300	112	900	35
400	90	1000	0
500	90		

Solution:

about 89,250 m²

Exercise:

Problem:

Choose the correct answer. When Simpson's rule is used to approximate the definite integral, it is necessary that the number of partitions be ____

- a. an even number
- b. odd number
- c. either an even or an odd number
- d. a multiple of 4

Exercise:

Problem: The "Simpson" sum is based on the area under a ____.

Solution:

parabola

Exercise:

Problem: The error formula for Simpson's rule depends on ____.

- a. $f(x)$
- b. $f'(x)$
- c. $f^{(4)}(x)$
- d. the number of steps

Glossary

absolute error

if B is an estimate of some quantity having an actual value of A , then the absolute error is given by $|A - B|$

midpoint rule

a rule that uses a Riemann sum of the form $M_n = \sum_{i=1}^n f(m_i) \Delta x$, where m_i is the midpoint of the i th subinterval to approximate $\int_a^b f(x) dx$

numerical integration

the variety of numerical methods used to estimate the value of a definite integral, including the midpoint rule, trapezoidal rule, and Simpson's rule

relative error

error as a percentage of the absolute value, given by $\left| \frac{A-B}{A} \right| = \left| \frac{A-B}{A} \right| \cdot 100\%$

Simpson's rule

a rule that approximates $\int_a^b f(x) dx$ using the integrals of a piecewise quadratic function. The approximation

S_n to $\int_a^b f(x) dx$ is given by $S_n = \frac{\Delta x}{3} \left(f(x_0) + 4f(x_1) + 2f(x_2) + 4f(x_3) + 2f(x_4) + 4f(x_5) + \dots + 2f(x_{n-2}) + 4f(x_{n-1}) + f(x_n) \right)$

trapezoidal rule a rule that approximates $\int_a^b f(x) dx$ using trapezoids

Improper Integrals

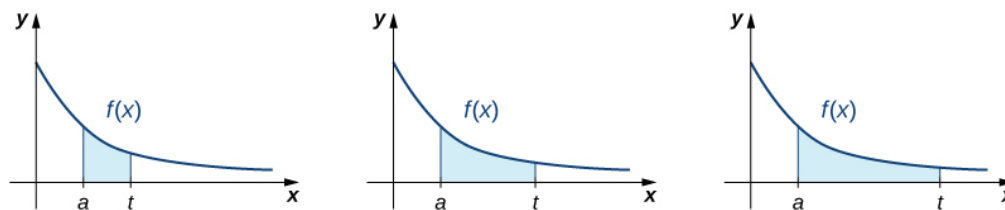
- Evaluate an integral over an infinite interval.
- Evaluate an integral over a closed interval with an infinite discontinuity within the interval.
- Use the comparison theorem to determine whether a definite integral is convergent.

Is the area between the graph of $f(x) = \frac{1}{x}$ and the x -axis over the interval $[1, +\infty)$ finite or infinite? If this same region is revolved about the x -axis, is the volume finite or infinite? Surprisingly, the area of the region described is infinite, but the volume of the solid obtained by revolving this region about the x -axis is finite.

In this section, we define integrals over an infinite interval as well as integrals of functions containing a discontinuity on the interval. Integrals of these types are called improper integrals. We examine several techniques for evaluating improper integrals, all of which involve taking limits.

Integrating over an Infinite Interval

How should we go about defining an integral of the type $\int_a^{+\infty} f(x)dx$? We can integrate $\int_a^t f(x)dx$ for any value of t , so it is reasonable to look at the behavior of this integral as we substitute larger values of t . [\[link\]](#) shows that $\int_a^t f(x)dx$ may be interpreted as area for various values of t . In other words, we may define an improper integral as a limit, taken as one of the limits of integration increases or decreases without bound.



To integrate a function over an infinite interval, we consider the limit of the integral as the upper limit increases without bound.

Note:

Definition

1. Let $f(x)$ be continuous over an interval of the form $[a, +\infty)$. Then

Equation:

$$\int_a^{+\infty} f(x)dx = \lim_{t \rightarrow +\infty} \int_a^t f(x)dx,$$

provided this limit exists.

2. Let $f(x)$ be continuous over an interval of the form $(-\infty, b]$. Then

Equation:

$$\int_{-\infty}^b f(x)dx = \lim_{t \rightarrow -\infty} \int_t^b f(x)dx,$$

provided this limit exists.

In each case, if the limit exists, then the **improper integral** is said to converge. If the limit does not exist, then the improper integral is said to diverge.

3. Let $f(x)$ be continuous over $(-\infty, +\infty)$. Then

Equation:

$$\int_{-\infty}^{+\infty} f(x)dx = \int_{-\infty}^0 f(x)dx + \int_0^{+\infty} f(x)dx,$$

provided that $\int_{-\infty}^0 f(x)dx$ and $\int_0^{+\infty} f(x)dx$ both converge. If either of these two integrals diverge, then $\int_{-\infty}^{+\infty} f(x)dx$ diverges. (It can be shown that, in fact, $\int_{-\infty}^{+\infty} f(x)dx = \int_{-\infty}^a f(x)dx + \int_a^{+\infty} f(x)dx$ for any value of a .)

In our first example, we return to the question we posed at the start of this section: Is the area between the graph of $f(x) = \frac{1}{x}$ and the x -axis over the interval $[1, +\infty)$ finite or infinite?

Example:

Exercise:

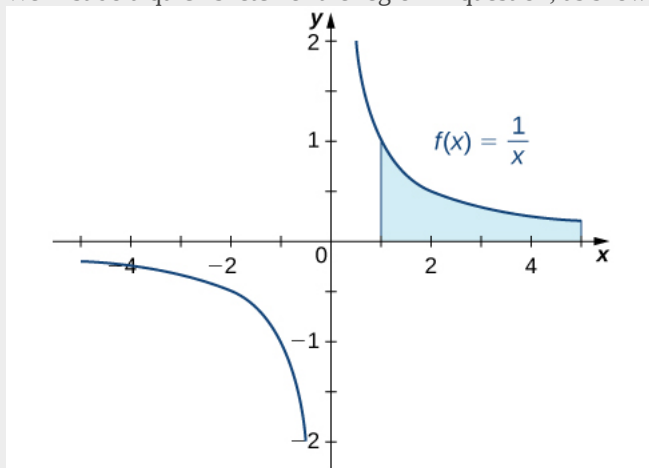
Problem:

Finding an Area

Determine whether the area between the graph of $f(x) = \frac{1}{x}$ and the x -axis over the interval $[1, +\infty)$ is finite or infinite.

Solution:

We first do a quick sketch of the region in question, as shown in the following graph.



We can find the area between the curve $f(x) = 1/x$ and the x -axis on an infinite interval.

We can see that the area of this region is given by $A = \int_1^{\infty} \frac{1}{x} dx$. Then we have

Equation:

$$\begin{aligned}
 A &= \int_1^{\infty} \frac{1}{x} dx \\
 &= \lim_{t \rightarrow +\infty} \int_1^t \frac{1}{x} dx && \text{Rewrite the improper integral as a limit.} \\
 &= \lim_{t \rightarrow +\infty} \ln |x| \bigg|_1^t && \text{Find the antiderivative.} \\
 &= \lim_{t \rightarrow +\infty} (\ln |t| - \ln 1) && \text{Evaluate the antiderivative.} \\
 &= +\infty. && \text{Evaluate the limit.}
 \end{aligned}$$

Since the improper integral diverges to $+\infty$, the area of the region is infinite.

Example:

Exercise:

Problem:

Finding a Volume

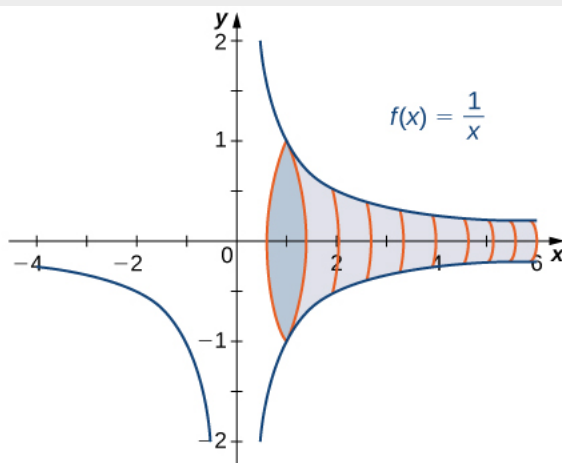
Find the volume of the solid obtained by revolving the region bounded by the graph of $f(x) = \frac{1}{x}$ and the x -axis over the interval $[1, +\infty)$ about the x -axis.

Solution:

The solid is shown in [\[link\]](#). Using the disk method, we see that the volume V is

Equation:

$$V = \pi \int_1^{+\infty} \frac{1}{x^2} dx.$$



The solid of revolution can be generated by rotating an infinite area about the x -axis.

Then we have

Equation:

$$\begin{aligned} V &= \pi \int_1^{+\infty} \frac{1}{x^2} dx \\ &= \pi \lim_{t \rightarrow +\infty} \int_1^t \frac{1}{x^2} dx && \text{Rewrite as a limit.} \\ &= \pi \lim_{t \rightarrow +\infty} \left. -\frac{1}{x} \right|_1^t && \text{Find the antiderivative.} \\ &= \pi \lim_{t \rightarrow +\infty} \left(-\frac{1}{t} + 1 \right) && \text{Evaluate the antiderivative.} \\ &= \pi. \end{aligned}$$

The improper integral converges to π . Therefore, the volume of the solid of revolution is π .

In conclusion, although the area of the region between the x -axis and the graph of $f(x) = 1/x$ over the interval $[1, +\infty)$ is infinite, the volume of the solid generated by revolving this region about the x -axis is finite. The solid generated is known as *Gabriel's Horn*.

Note:

Visit this [website](#) to read more about Gabriel's Horn.

Example:

Exercise:

Problem:

Chapter Opener: Traffic Accidents in a City



(credit: modification of work by
David McKelvey, Flickr)

In the chapter opener, we stated the following problem: Suppose that at a busy intersection, traffic accidents occur at an average rate of one every three months. After residents complained, changes were made to the traffic lights at the intersection. It has now been eight months since the changes were made and there have been no accidents. Were the changes effective or is the 8-month interval without an accident a result of chance?

Probability theory tells us that if the average time between events is k , the probability that X , the time between events, is between a and b is given by

Equation:

$$P(a \leq x \leq b) = \int_a^b f(x)dx \text{ where } f(x) = \begin{cases} 0 & \text{if } x < 0 \\ ke^{-kx} & \text{if } x \geq 0 \end{cases}.$$

Thus, if accidents are occurring at a rate of one every 3 months, then the probability that X , the time between accidents, is between a and b is given by

Equation:

$$P(a \leq x \leq b) = \int_a^b f(x)dx \text{ where } f(x) = \begin{cases} 0 & \text{if } x < 0 \\ 3e^{-3x} & \text{if } x \geq 0 \end{cases}.$$

To answer the question, we must compute $P(X \geq 8) = \int_8^{+\infty} 3e^{-3x}dx$ and decide whether it is likely that 8 months could have passed without an accident if there had been no improvement in the traffic situation.

Solution:

We need to calculate the probability as an improper integral:

Equation:

$$\begin{aligned} P(X \geq 8) &= \int_8^{+\infty} 3e^{-3x}dx \\ &= \lim_{t \rightarrow +\infty} \int_8^t 3e^{-3x}dx \\ &= \lim_{t \rightarrow +\infty} -e^{-3x} \Big|_8^t \\ &= \lim_{t \rightarrow +\infty} (-e^{-3t} + e^{-24}) \\ &\approx 3.8 \times 10^{-11}. \end{aligned}$$

The value 3.8×10^{-11} represents the probability of no accidents in 8 months under the initial conditions. Since this value is very, very small, it is reasonable to conclude the changes were effective.

Example:

Exercise:

Problem:

Evaluating an Improper Integral over an Infinite Interval

Evaluate $\int_{-\infty}^0 \frac{1}{x^2 + 4} dx$. State whether the improper integral converges or diverges.

Solution:

Begin by rewriting $\int_{-\infty}^0 \frac{1}{x^2 + 4} dx$ as a limit using [\[link\]](#) from the definition. Thus,

Equation:

$\int_{-\infty}^0 \frac{1}{x^2 + 4} dx = \lim_{x \rightarrow -\infty} \int_t^0 \frac{1}{x^2 + 4} dx$	Rewrite as a limit.
$= \lim_{t \rightarrow -\infty} \left. \frac{1}{2} \tan^{-1} \frac{x}{2} \right _t^0$	Find the antiderivative.
$= \frac{1}{2} \lim_{t \rightarrow -\infty} (\tan^{-1} 0 - \tan^{-1} \frac{t}{2})$	Evaluate the antiderivative.
$= \frac{\pi}{4}.$	Evaluate the limit and simplify.

The improper integral converges to $\frac{\pi}{4}$.

Example:**Exercise:****Problem:**

Evaluating an Improper Integral on $(-\infty, +\infty)$

Evaluate $\int_{-\infty}^{+\infty} x e^x dx$. State whether the improper integral converges or diverges.

Solution:

Start by splitting up the integral:

Equation:

$$\int_{-\infty}^{+\infty} x e^x dx = \int_{-\infty}^0 x e^x dx + \int_0^{+\infty} x e^x dx.$$

If either $\int_{-\infty}^0 x e^x dx$ or $\int_0^{+\infty} x e^x dx$ diverges, then $\int_{-\infty}^{+\infty} x e^x dx$ diverges. Compute each integral separately. For the first integral,

Equation:

$$\int_{-\infty}^0 xe^x dx = \lim_{t \rightarrow -\infty} \int_t^0 xe^x dx$$

$$= \lim_{t \rightarrow -\infty} (xe^x - e^x) \Big|_t^0$$

$$= \lim_{t \rightarrow -\infty} (-1 - te^t + e^t)$$

$$= -1.$$

Rewrite as a limit.

Use integration by parts to find the antiderivative. (Here $u = x$ and $dv = e^x_{dx}$.)

Evaluate the antiderivative.

Evaluate the limit. *Note:* $\lim_{t \rightarrow -\infty} te^t$ is

indeterminate of the form $0 \cdot \infty$. Thus,

$$\lim_{t \rightarrow -\infty} te^t = \lim_{t \rightarrow -\infty} \frac{t}{e^{-t}} = \lim_{t \rightarrow -\infty} \frac{-1}{e^{-t}} = \lim_{t \rightarrow -\infty} -e^t = 0 \text{ by}$$

L'Hôpital's Rule.

The first improper integral converges. For the second integral,

Equation:

$$\int_0^{+\infty} xe^x dx = \lim_{t \rightarrow +\infty} \int_0^t xe^x dx$$

$$= \lim_{t \rightarrow +\infty} (xe^x - e^x) \Big|_0^t$$

$$= \lim_{t \rightarrow +\infty} (te^t - e^t + 1)$$

$$= \lim_{t \rightarrow +\infty} ((t-1)e^t + 1)$$

$$= +\infty.$$

Rewrite as a limit.

Find the antiderivative.

Evaluate the antiderivative.

Rewrite. ($te^t - e^t$ is indeterminate.)

Evaluate the limit.

Thus, $\int_0^{+\infty} xe^x dx$ diverges. Since this integral diverges, $\int_{-\infty}^{+\infty} xe^x dx$ diverges as well.

Note:

Exercise:

Problem: Evaluate $\int_{-3}^{+\infty} e^{-x} dx$. State whether the improper integral converges or diverges.

Solution:

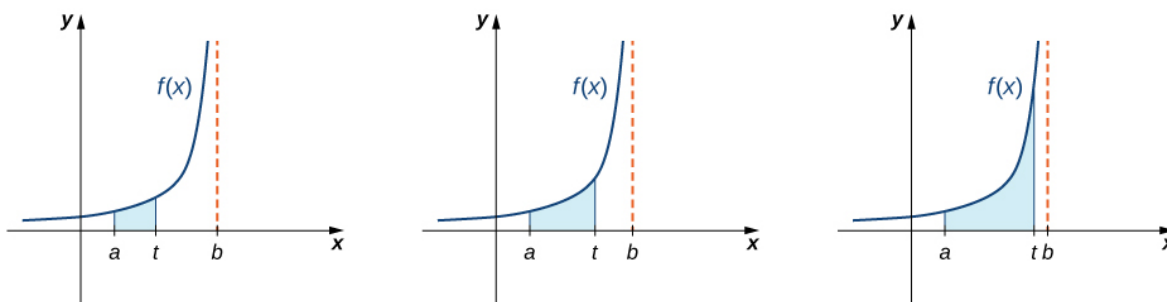
e^3 , converges

Hint

$$\int_{-3}^{+\infty} e^{-x} dx = \lim_{t \rightarrow +\infty} \int_{-3}^t e^{-x} dx$$

Integrating a Discontinuous Integrand

Now let's examine integrals of functions containing an infinite discontinuity in the interval over which the integration occurs. Consider an integral of the form $\int_a^b f(x)dx$, where $f(x)$ is continuous over $[a, b)$ and discontinuous at b . Since the function $f(x)$ is continuous over $[a, t]$ for all values of t satisfying $a < t < b$, the integral $\int_a^t f(x)dx$ is defined for all such values of t . Thus, it makes sense to consider the values of $\int_a^t f(x)dx$ as t approaches b for $a < t < b$. That is, we define $\int_a^b f(x)dx = \lim_{t \rightarrow b^-} \int_a^t f(x)dx$, provided this limit exists. [\[link\]](#) illustrates $\int_a^t f(x)dx$ as areas of regions for values of t approaching b .



As t approaches b from the left, the value of the area from a to t approaches the area from a to b .

We use a similar approach to define $\int_a^b f(x)dx$, where $f(x)$ is continuous over $(a, b]$ and discontinuous at a . We now proceed with a formal definition.

Note:

Definition

1. Let $f(x)$ be continuous over $[a, b)$. Then,

Equation:

$$\int_a^b f(x)dx = \lim_{t \rightarrow b^-} \int_a^t f(x)dx.$$

2. Let $f(x)$ be continuous over $(a, b]$. Then,

Equation:

$$\int_a^b f(x)dx = \lim_{t \rightarrow a^+} \int_t^b f(x)dx.$$

In each case, if the limit exists, then the improper integral is said to converge. If the limit does not exist, then the improper integral is said to diverge.

3. If $f(x)$ is continuous over $[a, b]$ except at a point c in (a, b) , then

Equation:

$$\int_a^b f(x)dx = \int_a^c f(x)dx + \int_c^b f(x)dx,$$

provided both $\int_a^c f(x)dx$ and $\int_c^b f(x)dx$ converge. If either of these integrals diverges, then $\int_a^b f(x)dx$ diverges.

The following examples demonstrate the application of this definition.

Example:

Exercise:

Problem:

Integrating a Discontinuous Integrand

Evaluate $\int_0^4 \frac{1}{\sqrt{4-x}} dx$, if possible. State whether the integral converges or diverges.

Solution:

The function $f(x) = \frac{1}{\sqrt{4-x}}$ is continuous over $[0, 4)$ and discontinuous at 4. Using [\[link\]](#) from the definition, rewrite $\int_0^4 \frac{1}{\sqrt{4-x}} dx$ as a limit:

Equation:

$$\begin{aligned} \int_0^4 \frac{1}{\sqrt{4-x}} dx &= \lim_{t \rightarrow 4^-} \int_0^t \frac{1}{\sqrt{4-x}} dx && \text{Rewrite as a limit.} \\ &= \lim_{t \rightarrow 4^-} \left(-2\sqrt{4-x} \right) \Big|_0^t && \text{Find the antiderivative.} \\ &= \lim_{t \rightarrow 4^-} \left(-2\sqrt{4-t} + 4 \right) && \text{Evaluate the antiderivative.} \\ &= 4. && \text{Evaluate the limit.} \end{aligned}$$

The improper integral converges.

Example:

Exercise:

Problem:

Integrating a Discontinuous Integrand

Evaluate $\int_0^2 x \ln x dx$. State whether the integral converges or diverges.

Solution:

Since $f(x) = x \ln x$ is continuous over $(0, 2]$ and is discontinuous at zero, we can rewrite the integral in limit form using [\[link\]](#):

Equation:

$$\begin{aligned}\int_0^2 x \ln x \, dx &= \lim_{t \rightarrow 0^+} \int_t^2 x \ln x \, dx \\ &= \lim_{t \rightarrow 0^+} \left(\frac{1}{2} x^2 \ln x - \frac{1}{4} x^2 \right) \bigg|_t^2 \\ &= \lim_{t \rightarrow 0^+} \left(2 \ln 2 - 1 - \frac{1}{2} t^2 \ln t + \frac{1}{4} t^2 \right). \\ &= 2 \ln 2 - 1.\end{aligned}$$

Rewrite as a limit.

Evaluate $\int x \ln x \, dx$ using integration by parts with $u = \ln x$ and $dv = x \, dx$.

Evaluate the antiderivative.

Evaluate the limit. $\lim_{t \rightarrow 0^+} t^2 \ln t$ is indeterminate.

To evaluate it, rewrite as a quotient and apply L'Hôpital's rule.

The improper integral converges.

Example:

Exercise:

Problem:

Integrating a Discontinuous Integrand

Evaluate $\int_{-1}^1 \frac{1}{x^3} \, dx$. State whether the improper integral converges or diverges.

Solution:

Since $f(x) = 1/x^3$ is discontinuous at zero, using [\[link\]](#), we can write

Equation:

$$\int_{-1}^1 \frac{1}{x^3} \, dx = \int_{-1}^0 \frac{1}{x^3} \, dx + \int_0^1 \frac{1}{x^3} \, dx.$$

If either of the two integrals diverges, then the original integral diverges. Begin with $\int_{-1}^0 \frac{1}{x^3} \, dx$:

Equation:

$$\begin{aligned}\int_{-1}^0 \frac{1}{x^3} \, dx &= \lim_{t \rightarrow 0^-} \int_{-1}^t \frac{1}{x^3} \, dx \\ &= \lim_{t \rightarrow 0^-} \left(-\frac{1}{2x^2} \right) \bigg|_{-1}^t \\ &= \lim_{t \rightarrow 0^-} \left(-\frac{1}{2t^2} + \frac{1}{2} \right) \\ &= +\infty.\end{aligned}$$

Rewrite as a limit.

Find the antiderivative.

Evaluate the antiderivative.

Evaluate the limit.

Therefore, $\int_{-1}^0 \frac{1}{x^3} dx$ diverges. Since $\int_{-1}^0 \frac{1}{x^3} dx$ diverges, $\int_{-1}^1 \frac{1}{x^3} dx$ diverges.

Note:

Exercise:

Problem: Evaluate $\int_0^2 \frac{1}{x} dx$. State whether the integral converges or diverges.

Solution:

$+\infty$, diverges

Hint

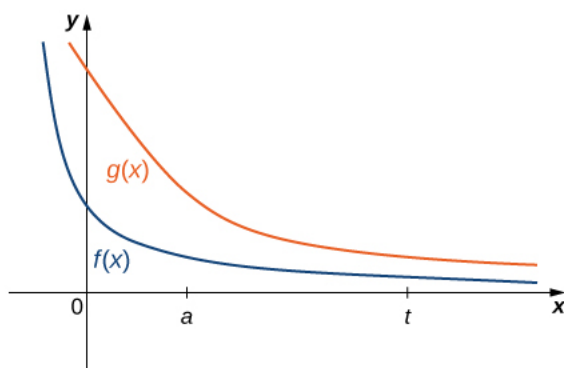
Write $\int_0^2 \frac{1}{x} dx$ in limit form using [\[link\]](#).

A Comparison Theorem

It is not always easy or even possible to evaluate an improper integral directly; however, by comparing it with another carefully chosen integral, it may be possible to determine its convergence or divergence. To see this, consider two continuous functions $f(x)$ and $g(x)$ satisfying $0 \leq f(x) \leq g(x)$ for $x \geq a$ ([\[link\]](#)). In this case, we may view integrals of these functions over intervals of the form $[a, t]$ as areas, so we have the relationship

Equation:

$$0 \leq \int_a^t f(x) dx \leq \int_a^t g(x) dx \text{ for } t \geq a.$$



If $0 \leq f(x) \leq g(x)$ for $x \geq a$, then for $t \geq a$,

$$\int_a^t f(x) dx \leq \int_a^t g(x) dx.$$

Thus, if

Equation:

$$\int_a^{+\infty} f(x)dx = \lim_{t \rightarrow +\infty} \int_a^t f(x)dx = +\infty,$$

then

$\int_a^{+\infty} g(x)dx = \lim_{t \rightarrow +\infty} \int_a^t g(x)dx = +\infty$ as well. That is, if the area of the region between the graph of $f(x)$ and the x -axis over $[a, +\infty)$ is infinite, then the area of the region between the graph of $g(x)$ and the x -axis over $[a, +\infty)$ is infinite too.

On the other hand, if

$\int_a^{+\infty} g(x)dx = \lim_{t \rightarrow +\infty} \int_a^t g(x)dx = L$ for some real number L , then

$\int_a^{+\infty} f(x)dx = \lim_{t \rightarrow +\infty} \int_a^t f(x)dx$ must converge to some value less than or equal to L , since $\int_a^t f(x)dx$ increases as t increases and $\int_a^t f(x)dx \leq L$ for all $t \geq a$.

If the area of the region between the graph of $g(x)$ and the x -axis over $[a, +\infty)$ is finite, then the area of the region between the graph of $f(x)$ and the x -axis over $[a, +\infty)$ is also finite.

These conclusions are summarized in the following theorem.

Note:

A Comparison Theorem

Let $f(x)$ and $g(x)$ be continuous over $[a, +\infty)$. Assume that $0 \leq f(x) \leq g(x)$ for $x \geq a$.

i. If $\int_a^{+\infty} f(x)dx = \lim_{t \rightarrow +\infty} \int_a^t f(x)dx = +\infty$, then $\int_a^{+\infty} g(x)dx = \lim_{t \rightarrow +\infty} \int_a^t g(x)dx = +\infty$.

ii. If $\int_a^{+\infty} g(x)dx = \lim_{t \rightarrow +\infty} \int_a^t g(x)dx = L$, where L is a real number, then

$\int_a^{+\infty} f(x)dx = \lim_{t \rightarrow +\infty} \int_a^t f(x)dx = M$ for some real number $M \leq L$.

Example:

Exercise:

Problem:

Applying the Comparison Theorem

Use a comparison to show that $\int_1^{+\infty} \frac{1}{xe^x} dx$ converges.

Solution:

We can see that

Equation:

$$0 \leq \frac{1}{xe^x} \leq \frac{1}{e^x} = e^{-x},$$

so if $\int_1^{+\infty} e^{-x} dx$ converges, then so does $\int_1^{+\infty} \frac{1}{xe^x} dx$. To evaluate $\int_1^{+\infty} e^{-x} dx$, first rewrite it as a limit:

Equation:

$$\begin{aligned} \int_1^{+\infty} e^{-x} dx &= \lim_{t \rightarrow +\infty} \int_1^t e^{-x} dx \\ &= \lim_{t \rightarrow +\infty} (-e^{-x}) \Big|_1^t \\ &= \lim_{t \rightarrow +\infty} (-e^{-t} + e^{-1}) \\ &= e^{-1}. \end{aligned}$$

Since $\int_1^{+\infty} e^{-x} dx$ converges, so does $\int_1^{+\infty} \frac{1}{xe^x} dx$.

Example:

Exercise:

Problem:

Applying the Comparison Theorem

Use the comparison theorem to show that $\int_1^{+\infty} \frac{1}{x^p} dx$ diverges for all $p < 1$.

Solution:

For $p < 1$, $1/x \leq 1/(x^p)$ over $[1, +\infty)$. In [\[link\]](#), we showed that $\int_1^{+\infty} \frac{1}{x} dx = +\infty$. Therefore,

$\int_1^{+\infty} \frac{1}{x^p} dx$ diverges for all $p < 1$.

Note:

Exercise:

Problem: Use a comparison to show that $\int_e^{+\infty} \frac{\ln x}{x} dx$ diverges.

Solution:

Since $\int_e^{+\infty} \frac{1}{x} dx = +\infty$, $\int_e^{+\infty} \frac{\ln x}{x} dx$ diverges.

Hint

$$\frac{1}{x} \leq \frac{\ln x}{x} \text{ on } [e, +\infty)$$

Note:**Laplace Transforms**

In the last few chapters, we have looked at several ways to use integration for solving real-world problems. For this next project, we are going to explore a more advanced application of integration: integral transforms. Specifically, we describe the Laplace transform and some of its properties. The Laplace transform is used in engineering and physics to simplify the computations needed to solve some problems. It takes functions expressed in terms of time and *transforms* them to functions expressed in terms of frequency. It turns out that, in many cases, the computations needed to solve problems in the frequency domain are much simpler than those required in the time domain.

The Laplace transform is defined in terms of an integral as

Equation:

$$L\{f(t)\} = F(s) = \int_0^{\infty} e^{-st} f(t) dt.$$

Note that the input to a Laplace transform is a function of time, $f(t)$, and the output is a function of frequency, $F(s)$. Although many real-world examples require the use of complex numbers (involving the imaginary number $i = \sqrt{-1}$), in this project we limit ourselves to functions of real numbers.

Let's start with a simple example. Here we calculate the Laplace transform of $f(t) = t$. We have

Equation:

$$L\{t\} = \int_0^{\infty} te^{-st} dt.$$

This is an improper integral, so we express it in terms of a limit, which gives

Equation:

$$L\{t\} = \int_0^{\infty} te^{-st} dt = \lim_{z \rightarrow \infty} \int_0^z te^{-st} dt.$$

Now we use integration by parts to evaluate the integral. Note that we are integrating with respect to t , so we treat the variable s as a constant. We have

Equation:

$$\begin{aligned} u &= t & dv &= e^{-st} dt \\ du &= dt & v &= -\frac{1}{s} e^{-st}. \end{aligned}$$

Then we obtain

Equation:

$$\begin{aligned}
\lim_{z \rightarrow \infty} \int_0^z t e^{-st} dt &= \lim_{z \rightarrow \infty} \left[\left[-\frac{t}{s} e^{-st} \right] \Big|_0^z + \frac{1}{s} \int_0^z e^{-st} dt \right] \\
&= \lim_{z \rightarrow \infty} \left[\left[-\frac{z}{s} e^{-sz} + \frac{0}{s} e^{-0s} \right] + \frac{1}{s} \int_0^z e^{-st} dt \right] \\
&= \lim_{z \rightarrow \infty} \left[\left[-\frac{z}{s} e^{-sz} + 0 \right] - \frac{1}{s} \left[\frac{e^{-st}}{s} \right] \Big|_0^z \right] \\
&= \lim_{z \rightarrow \infty} \left[\left[-\frac{z}{s} e^{-sz} \right] - \frac{1}{s^2} [e^{-sz} - 1] \right] \\
&= \lim_{z \rightarrow \infty} \left[-\frac{z}{s e^{sz}} \right] - \lim_{z \rightarrow \infty} \left[\frac{1}{s^2 e^{sz}} \right] + \lim_{z \rightarrow \infty} \frac{1}{s^2} \\
&= 0 - 0 + \frac{1}{s^2} \\
&= \frac{1}{s^2}.
\end{aligned}$$

1. Calculate the Laplace transform of $f(t) = 1$.
2. Calculate the Laplace transform of $f(t) = e^{-3t}$.
3. Calculate the Laplace transform of $f(t) = t^2$. (Note, you will have to integrate by parts twice.)

Laplace transforms are often used to solve differential equations. Differential equations are not covered in detail until later in this book; but, for now, let's look at the relationship between the Laplace transform of a function and the Laplace transform of its derivative.

Let's start with the definition of the Laplace transform. We have

Equation:

$$L\{f(t)\} = \int_0^\infty e^{-st} f(t) dt = \lim_{z \rightarrow \infty} \int_0^z e^{-st} f(t) dt.$$

4. Use integration by parts to evaluate $\lim_{z \rightarrow \infty} \int_0^z e^{-st} f(t) dt$. (Let $u = f(t)$ and $dv = e^{-st} dt$.)

After integrating by parts and evaluating the limit, you should see that

Equation:

$$L\{f(t)\} = \frac{f(0)}{s} + \frac{1}{s} [L\{f'(t)\}].$$

Then,

Equation:

$$L\{f'(t)\} = sL\{f(t)\} - f(0).$$

Thus, differentiation in the time domain simplifies to multiplication by s in the frequency domain.

The final thing we look at in this project is how the Laplace transforms of $f(t)$ and its antiderivative are

related. Let $g(t) = \int_0^t f(u) du$. Then,

Equation:

$$L\{g(t)\} = \int_0^\infty e^{-st} g(t) dt = \lim_{z \rightarrow \infty} \int_0^z e^{-st} g(t) dt.$$

5. Use integration by parts to evaluate $\lim_{z \rightarrow \infty} \int_0^z e^{-st} g(t) dt$. (Let $u = g(t)$ and $dv = e^{-st} dt$. Note, by the way, that we have defined $g(t)$, $du = f(t) dt$.)

As you might expect, you should see that

Equation:

$$L\{g(t)\} = \frac{1}{s} \cdot L\{f(t)\}.$$

Integration in the time domain simplifies to division by s in the frequency domain.

Key Concepts

- Integrals of functions over infinite intervals are defined in terms of limits.
- Integrals of functions over an interval for which the function has a discontinuity at an endpoint may be defined in terms of limits.
- The convergence or divergence of an improper integral may be determined by comparing it with the value of an improper integral for which the convergence or divergence is known.

Key Equations

- **Improper integrals**

$$\int_a^{+\infty} f(x)dx = \lim_{t \rightarrow +\infty} \int_a^t f(x)dx$$

$$\int_{-\infty}^b f(x)dx = \lim_{t \rightarrow -\infty} \int_t^b f(x)dx$$

$$\int_{-\infty}^{+\infty} f(x)dx = \int_{-\infty}^0 f(x)dx + \int_0^{+\infty} f(x)dx$$

Evaluate the following integrals. If the integral is not convergent, answer “divergent.”

Exercise:

Problem: $\int_2^4 \frac{dx}{(x-3)^2}$

Solution:

divergent

Exercise:

Problem: $\int_0^{\infty} \frac{1}{4+x^2} dx$

Exercise:

Problem: $\int_0^2 \frac{1}{\sqrt{4-x^2}} dx$

Solution:

$$\frac{\pi}{2}$$

Exercise:

Problem: $\int_1^{\infty} \frac{1}{x \ln x} dx$

Exercise:

Problem: $\int_1^{\infty} x e^{-x} dx$

Solution:

$$\frac{2}{e}$$

Exercise:

Problem: $\int_{-\infty}^{\infty} \frac{x}{x^2 + 1} dx$

Exercise:

Problem:

Without integrating, determine whether the integral $\int_1^{\infty} \frac{1}{\sqrt{x^3 + 1}} dx$ converges or diverges by comparing the function $f(x) = \frac{1}{\sqrt{x^3 + 1}}$ with $g(x) = \frac{1}{\sqrt{x^3}}$.

Solution:

Converges

Exercise:

Problem: Without integrating, determine whether the integral $\int_1^{\infty} \frac{1}{\sqrt{x + 1}} dx$ converges or diverges.

Determine whether the improper integrals converge or diverge. If possible, determine the value of the integrals that converge.

Exercise:

Problem: $\int_0^{\infty} e^{-x} \cos x dx$

Solution:

Converges to 1/2.

Exercise:

Problem: $\int_1^{\infty} \frac{\ln x}{x} dx$

Exercise:

Problem: $\int_0^1 \frac{\ln x}{\sqrt{x}} dx$

Solution:

-4

Exercise:

Problem: $\int_0^1 \ln x \, dx$

Exercise:

Problem: $\int_{-\infty}^{\infty} \frac{1}{x^2 + 1} dx$

Solution:

π

Exercise:

Problem: $\int_1^5 \frac{dx}{\sqrt{x-1}}$

Exercise:

Problem: $\int_{-2}^2 \frac{dx}{(1+x)^2}$

Solution:

diverges

Exercise:

Problem: $\int_0^{\infty} e^{-x} dx$

Exercise:

Problem: $\int_0^{\infty} \sin x \, dx$

Solution:

diverges

Exercise:

Problem: $\int_{-\infty}^{\infty} \frac{e^x}{1 + e^{2x}} dx$

Exercise:

Problem: $\int_0^1 \frac{dx}{\sqrt[3]{x}}$

Solution:

1.5

Exercise:

Problem: $\int_0^2 \frac{dx}{x^3}$

Exercise:

Problem: $\int_{-1}^2 \frac{dx}{x^3}$

Solution:

diverges

Exercise:

Problem: $\int_0^1 \frac{dx}{\sqrt{1-x^2}}$

Exercise:

Problem: $\int_0^3 \frac{1}{x-1} dx$

Solution:

diverges

Exercise:

Problem: $\int_1^\infty \frac{5}{x^3} dx$

Exercise:

Problem: $\int_3^5 \frac{5}{(x-4)^2} dx$

Solution:

diverges

Determine the convergence of each of the following integrals by comparison with the given integral. If the integral converges, find the number to which it converges.

Exercise:

Problem: $\int_1^\infty \frac{dx}{x^2 + 4x}$; compare with $\int_1^\infty \frac{dx}{x^2}$.

Exercise:

Problem: $\int_1^{\infty} \frac{dx}{\sqrt{x} + 1}$; compare with $\int_1^{\infty} \frac{dx}{2\sqrt{x}}$.

Solution:

Both integrals diverge.

Evaluate the integrals. If the integral diverges, answer “diverges.”

Exercise:

Problem: $\int_1^{\infty} \frac{dx}{x^e}$

Exercise:

Problem: $\int_0^1 \frac{dx}{x^{\pi}}$

Solution:

diverges

Exercise:

Problem: $\int_0^1 \frac{dx}{\sqrt{1-x}}$

Exercise:

Problem: $\int_0^1 \frac{dx}{1-x}$

Solution:

diverges

Exercise:

Problem: $\int_{-\infty}^0 \frac{dx}{x^2 + 1}$

Exercise:

Problem: $\int_{-1}^1 \frac{dx}{\sqrt{1-x^2}}$

Solution:

π

Exercise:

Problem: $\int_0^1 \frac{\ln x}{x} dx$

Exercise:

Problem: $\int_0^e \ln(x) dx$

Solution:

0.0

Exercise:

Problem: $\int_0^\infty x e^{-x} dx$

Exercise:

Problem: $\int_{-\infty}^\infty \frac{x}{(x^2 + 1)^2} dx$

Solution:

0.0

Exercise:

Problem: $\int_0^\infty e^{-x} dx$

Evaluate the improper integrals. Each of these integrals has an infinite discontinuity either at an endpoint or at an interior point of the interval.

Exercise:

Problem: $\int_0^9 \frac{dx}{\sqrt{9-x}}$

Solution:

6.0

Exercise:

Problem: $\int_{-27}^1 \frac{dx}{x^{2/3}}$

Exercise:

Problem: $\int_0^3 \frac{dx}{\sqrt{9-x^2}}$

Solution:

$$\frac{\pi}{2}$$

Exercise:

Problem: $\int_6^{24} \frac{dt}{t\sqrt{t^2 - 36}}$

Exercise:

Problem: $\int_0^4 x \ln(4x) dx$

Solution:

$$8 \ln(16) - 4$$

Exercise:

Problem: $\int_0^3 \frac{x}{\sqrt{9 - x^2}} dx$

Exercise:

Problem: Evaluate $\int_{.5}^1 \frac{dx}{\sqrt{1 - x^2}}$. (Be careful!) (Express your answer using three decimal places.)

Solution:

$$1.047$$

Exercise:

Problem: Evaluate $\int_1^4 \frac{dx}{\sqrt{x^2 - 1}}$. (Express the answer in exact form.)

Exercise:

Problem: Evaluate $\int_2^\infty \frac{dx}{(x^2 - 1)^{3/2}}$.

Solution:

$$-1 + \frac{2}{\sqrt{3}}$$

Exercise:

Problem: Find the area of the region in the first quadrant between the curve $y = e^{-6x}$ and the x -axis.

Exercise:

Problem: Find the area of the region bounded by the curve $y = \frac{7}{x^2}$, the x -axis, and on the left by $x = 1$.

Solution:

$$7.0$$

Exercise:

Problem: Find the area under the curve $y = \frac{1}{(x+1)^{3/2}}$, bounded on the left by $x = 3$.

Exercise:

Problem: Find the area under $y = \frac{5}{1+x^2}$ in the first quadrant.

Solution:

$$\frac{5\pi}{2}$$

Exercise:**Problem:**

Find the volume of the solid generated by revolving about the x -axis the region under the curve $y = \frac{3}{x}$ from $x = 1$ to $x = \infty$.

Exercise:**Problem:**

Find the volume of the solid generated by revolving about the y -axis the region under the curve $y = 6e^{-2x}$ in the first quadrant.

Solution:

$$3\pi$$

Exercise:**Problem:**

Find the volume of the solid generated by revolving about the x -axis the area under the curve $y = 3e^{-x}$ in the first quadrant.

The Laplace transform of a continuous function over the interval $[0, \infty)$ is defined by $F(s) = \int_0^\infty e^{-sx} f(x) dx$ (see the Student Project). This definition is used to solve some important initial-value problems in differential equations, as discussed later. The domain of F is the set of all real numbers s such that the improper integral converges. Find the Laplace transform F of each of the following functions and give the domain of F .

Exercise:

Problem: $f(x) = 1$

Solution:

$$\frac{1}{s}, s > 0$$

Exercise:

Problem: $f(x) = x$

Exercise:

Problem: $f(x) = \cos(2x)$

Solution:

$$\frac{s}{s^2+4}, s > 0$$

Exercise:

Problem: $f(x) = e^{ax}$

Exercise:

Problem: Use the formula for arc length to show that the circumference of the circle $x^2 + y^2 = 1$ is 2π .

Solution:

Answers will vary.

A non-negative function is a probability density function if it satisfies the following definition: $\int_{-\infty}^{\infty} f(t)dt = 1$.

The probability that a random variable x lies between a and b is given by $P(a \leq x \leq b) = \int_a^b f(t)dt$.

Exercise:

Problem: Show that $f(x) = \begin{cases} 0 & \text{if } x < 0 \\ 7e^{-7x} & \text{if } x \geq 0 \end{cases}$ is a probability density function.

Exercise:

Problem:

Find the probability that x is between 0 and 0.3. (Use the function defined in the preceding problem.) Use four-place decimal accuracy.

Solution:

0.8775

Chapter Review Exercises

For the following exercises, determine whether the statement is true or false. Justify your answer with a proof or a counterexample.

Exercise:

Problem: $\int e^x \sin(x) dx$ cannot be integrated by parts.

Exercise:

Problem: $\int \frac{1}{x^4 + 1} dx$ cannot be integrated using partial fractions.

Solution:

False

Exercise:

Problem: In numerical integration, increasing the number of points decreases the error.

Exercise:

Problem: Integration by parts can always yield the integral.

Solution:

False

For the following exercises, evaluate the integral using the specified method.

Exercise:

Problem: $\int x^2 \sin(4x) dx$ using integration by parts

Exercise:

Problem: $\int \frac{1}{x^2 \sqrt{x^2 + 16}} dx$ using trigonometric substitution

Solution:

$$-\frac{\sqrt{x^2+16}}{16x} + C$$

Exercise:

Problem: $\int \sqrt{x} \ln(x) dx$ using integration by parts

Exercise:

Problem: $\int \frac{3x}{x^3 + 2x^2 - 5x - 6} dx$ using partial fractions

Solution:

$$\frac{1}{10} (4 \ln(2 - x) + 5 \ln(x + 1) - 9 \ln(x + 3)) + C$$

Exercise:

Problem: $\int \frac{x^5}{(4x^2 + 4)^{5/2}} dx$ using trigonometric substitution

Exercise:

Problem: $\int \frac{\sqrt{4 - \sin^2(x)}}{\sin^2(x)} \cos(x) dx$ using a table of integrals or a CAS

Solution:

$$-\frac{\sqrt{4 - \sin^2(x)}}{\sin(x)} - \frac{x}{2} + C$$

For the following exercises, integrate using whatever method you choose.

Exercise:

Problem: $\int \sin^2(x) \cos^2(x) dx$

Exercise:

Problem: $\int x^3 \sqrt{x^2 + 2} dx$

Solution:

$$\frac{1}{15} (x^2 + 2)^{3/2} (3x^2 - 4) + C$$

Exercise:

Problem: $\int \frac{3x^2 + 1}{x^4 - 2x^3 - x^2 + 2x} dx$

Exercise:

Problem: $\int \frac{1}{x^4 + 4} dx$

Solution:

$$\frac{1}{16} \ln \left(\frac{x^2 + 2x + 2}{x^2 - 2x + 2} \right) - \frac{1}{8} \tan^{-1}(1 - x) + \frac{1}{8} \tan^{-1}(x + 1) + C$$

Exercise:

Problem: $\int \frac{\sqrt{3 + 16x^4}}{x^4} dx$

For the following exercises, approximate the integrals using the midpoint rule, trapezoidal rule, and Simpson's rule using four subintervals, rounding to three decimals.

Exercise:

Problem: [T] $\int_1^2 \sqrt{x^5 + 2} dx$

Solution:

$$M_4 = 3.312, T_4 = 3.354, S_4 = 3.326$$

Exercise:

Problem: [T] $\int_0^{\sqrt{\pi}} e^{-\sin(x^2)} dx$

Exercise:

Problem: [T] $\int_1^4 \frac{\ln(1/x)}{x} dx$

Solution:

$$M_4 = -0.982, T_4 = -0.917, S_4 = -0.952$$

For the following exercises, evaluate the integrals, if possible.

Exercise:

Problem: $\int_1^\infty \frac{1}{x^n} dx$, for what values of n does this integral converge or diverge?

Exercise:

Problem: $\int_1^\infty \frac{e^{-x}}{x} dx$

Solution:

approximately 0.2194

For the following exercises, consider the gamma function given by $\Gamma(a) = \int_0^\infty e^{-y} y^{a-1} dy$.

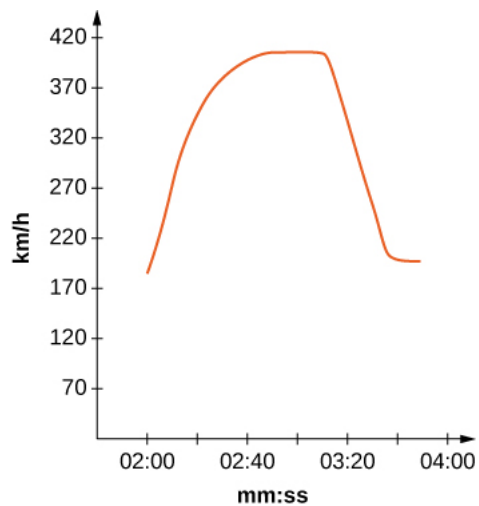
Exercise:

Problem: Show that $\Gamma(a) = (a-1)\Gamma(a-1)$.

Exercise:

Problem: Extend to show that $\Gamma(a) = (a-1)!$, assuming a is a positive integer.

The fastest car in the world, the Bugati Veyron, can reach a top speed of 408 km/h. The graph represents its velocity.



Exercise:

Problem:

[T] Use the graph to estimate the velocity every 20 sec and fit to a graph of the form $v(t) = a \exp^{bx} \sin(cx) + d$. (*Hint: Consider the time units.*)

Exercise:**Problem:**

[T] Using your function from the previous problem, find exactly how far the Bugati Veyron traveled in the 1 min 40 sec included in the graph.

Solution:

Answers may vary. Ex: 9.405 km

Glossary**improper integral**

an integral over an infinite interval or an integral of a function containing an infinite discontinuity on the interval; an improper integral is defined in terms of a limit. The improper integral converges if this limit is a finite real number; otherwise, the improper integral diverges

Basics of Differential Equations

- Identify the order of a differential equation.
- Explain what is meant by a solution to a differential equation.
- Distinguish between the general solution and a particular solution of a differential equation.
- Identify an initial-value problem.
- Identify whether a given function is a solution to a differential equation or an initial-value problem.

Calculus is the mathematics of change, and rates of change are expressed by derivatives. Thus, one of the most common ways to use calculus is to set up an equation containing an unknown function $y = f(x)$ and its derivative, known as a *differential equation*. Solving such equations often provides information about how quantities change and frequently provides insight into how and why the changes occur.

Techniques for solving differential equations can take many different forms, including direct solution, use of graphs, or computer calculations. We introduce the main ideas in this chapter and describe them in a little more detail later in the course. In this section we study what differential equations are, how to verify their solutions, some methods that are used for solving them, and some examples of common and useful equations.

General Differential Equations

Consider the equation $y' = 3x^2$, which is an example of a differential equation because it includes a derivative. There is a relationship between the variables x and y : y is an unknown function of x . Furthermore, the left-hand side of the equation is the derivative of y . Therefore we can interpret this equation as follows: Start with some function $y = f(x)$ and take its derivative. The answer must be equal to $3x^2$. What function has a derivative that is equal to $3x^2$? One such function is $y = x^3$, so this function is considered a **solution to a differential equation**.

Note:**Definition**

A **differential equation** is an equation involving an unknown function $y = f(x)$ and one or more of its derivatives. A solution to a differential equation is a function $y = f(x)$ that satisfies the differential equation when f and its derivatives are substituted into the equation.

Note:

Go to this [website](#) to explore more on this topic.

Some examples of differential equations and their solutions appear in [\[link\]](#).

Equation	Solution
$y' = 2x$	$y = x^2$
$y' + 3y = 6x + 11$	$y = e^{-3x} + 2x + 3$
$y'' - 3y' + 2y = 24e^{-2x}$	$y = 3e^x - 4e^{2x} + 2e^{-2x}$

Examples of Differential Equations and Their Solutions

Note that a solution to a differential equation is not necessarily unique, primarily because the derivative of a constant is zero. For example, $y = x^2 + 4$ is also a solution to the first differential equation in [\[link\]](#). We will return to this idea a little bit later in this section. For now, let's focus on what it means for a function to be a solution to a differential equation.

Example:

Exercise:

Problem:

Verifying Solutions of Differential Equations

Verify that the function $y = e^{-3x} + 2x + 3$ is a solution to the differential equation $y' + 3y = 6x + 11$.

Solution:

To verify the solution, we first calculate y' using the chain rule for derivatives. This gives $y' = -3e^{-3x} + 2$. Next we substitute y and y' into the left-hand side of the differential equation:

Equation:

$$(-3e^{-2x} + 2) + 3(e^{-2x} + 2x + 3).$$

The resulting expression can be simplified by first distributing to eliminate the parentheses, giving

Equation:

$$-3e^{-2x} + 2 + 3e^{-2x} + 6x + 9.$$

Combining like terms leads to the expression $6x + 11$, which is equal to the right-hand side of the differential equation. This result verifies that $y = e^{-3x} + 2x + 3$ is a solution of the differential equation.

Note:

Exercise:

Problem:

Verify that $y = 2e^{3x} - 2x - 2$ is a solution to the differential equation $y' - 3y = 6x + 4$.

Hint

First calculate y' then substitute both y' and y into the left-hand side.

It is convenient to define characteristics of differential equations that make it easier to talk about them and categorize them. The most basic characteristic of a differential equation is its order.

Note:

Definition

The **order of a differential equation** is the highest order of any derivative of the unknown function that appears in the equation.

Example:

Exercise:

Problem:

Identifying the Order of a Differential Equation

What is the order of each of the following differential equations?

a. $y' - 4y = x^2 - 3x + 4$

b. $x^2 y''' - 3xy'' + xy' - 3y = \sin x$

c. $\frac{4}{x} y^{(4)} - \frac{6}{x^2} y'' + \frac{12}{x^4} y = x^3 - 3x^2 + 4x - 12$

Solution:

- a. The highest derivative in the equation is y' , so the order is 1.
- b. The highest derivative in the equation is y''' , so the order is 3.
- c. The highest derivative in the equation is $y^{(4)}$, so the order is 4.

Note:

Exercise:

Problem: What is the order of the following differential equation?

Equation:

$$(x^4 - 3x)y^{(5)} - (3x^2 + 1)y' + 3y = \sin x \cos x$$

Solution:

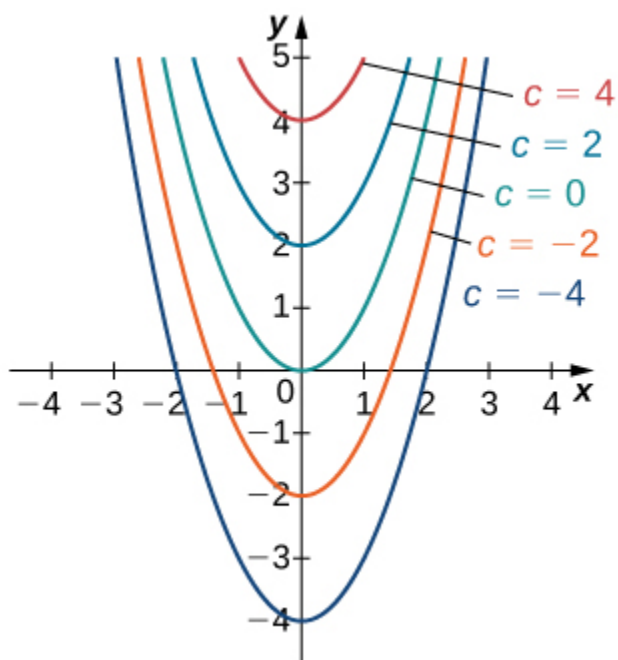
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Hint

What is the highest derivative in the equation?

General and Particular Solutions

We already noted that the differential equation $y' = 2x$ has at least two solutions: $y = x^2$ and $y = x^2 + 4$. The only difference between these two solutions is the last term, which is a constant. What if the last term is a different constant? Will this expression still be a solution to the differential equation? In fact, any function of the form $y = x^2 + C$, where C represents any constant, is a solution as well. The reason is that the derivative of $x^2 + C$ is $2x$, regardless of the value of C . It can be shown that any solution of this differential equation must be of the form $y = x^2 + C$. This is an example of a **general solution** to a differential equation. A graph of some of these solutions is given in [\[link\]](#). (Note: in this graph we used even integer values for C ranging between -4 and 4 . In fact, there is no restriction on the value of C ; it can be an integer or not.)



Family of solutions to the differential equation $y' = 2x$.

In this example, we are free to choose any solution we wish; for example, $y = x^2 - 3$ is a member of the family of solutions to this differential equation. This is called a **particular solution** to the differential equation. A particular solution can often be uniquely identified if we are given additional information about the problem.

Example:

Exercise:

Problem:

Finding a Particular Solution

Find the particular solution to the differential equation $y' = 2x$ passing through the point $(2, 7)$.

Solution:

Any function of the form $y = x^2 + C$ is a solution to this differential equation. To determine the value of C , we substitute the values $x = 2$ and $y = 7$ into this equation and solve for C :

Equation:

$$\begin{aligned}y &= x^2 + C \\7 &= 2^2 + C = 4 + C \\C &= 3.\end{aligned}$$

Therefore the particular solution passing through the point $(2, 7)$ is $y = x^2 + 3$.

Note:**Exercise:**

Problem: Find the particular solution to the differential equation

Equation:

$$y' = 4x + 3$$

passing through the point $(1, 7)$, given that $y = 2x^2 + 3x + C$ is a general solution to the differential equation.

Solution:

$$y = 2x^2 + 3x + 2$$

Hint

First substitute $x = 1$ and $y = 7$ into the equation, then solve for C .

Initial-Value Problems

Usually a given differential equation has an infinite number of solutions, so it is natural to ask which one we want to use. To choose one solution, more information is needed. Some specific information that can be useful is an **initial value**, which is an ordered pair that is used to find a particular solution.

A differential equation together with one or more initial values is called an **initial-value problem**. The general rule is that the number of initial values needed for an initial-value problem is equal to the order of the differential equation. For example, if we have the differential equation $y' = 2x$, then $y(3) = 7$ is an initial value, and when taken together, these equations form an initial-value problem. The differential equation $y'' - 3y' + 2y = 4e^x$ is second order, so we need two initial values. With initial-value problems of order greater than one, the same value should be used for the independent variable. An example of initial values for this second-order equation would be $y(0) = 2$ and $y'(0) = -1$. These two initial values together with the differential equation form an initial-value problem. These problems are so named because often the independent variable in the unknown function is t , which represents time. Thus, a value of $t = 0$ represents the beginning of the problem.

Example:

Exercise:

Problem:

Verifying a Solution to an Initial-Value Problem

Verify that the function $y = 2e^{-2t} + e^t$ is a solution to the initial-value problem

Equation:

$$y' + 2y = 3e^t, \quad y(0) = 3.$$

Solution:

For a function to satisfy an initial-value problem, it must satisfy both the differential equation and the initial condition. To show that y satisfies the differential equation, we start by calculating y' . This gives $y' = -4e^{-2t} + e^t$. Next we substitute both y and y' into the left-hand side of the differential equation and simplify:

Equation:

$$\begin{aligned}y' + 2y &= (-4e^{-2t} + e^t) + 2(2e^{-2t} + e^t) \\&= -4e^{-2t} + e^t + 4e^{-2t} + 2e^t \\&= 3e^t.\end{aligned}$$

This is equal to the right-hand side of the differential equation, so $y = 2e^{-2t} + e^t$ solves the differential equation. Next we calculate $y(0)$:

Equation:

$$\begin{aligned}y(0) &= 2e^{-2(0)} + e^0 \\&= 2 + 1 \\&= 3.\end{aligned}$$

This result verifies the initial value. Therefore the given function satisfies the initial-value problem.

Note:**Exercise:****Problem:**

Verify that $y = 3e^{2t} + 4\sin t$ is a solution to the initial-value problem

Equation:

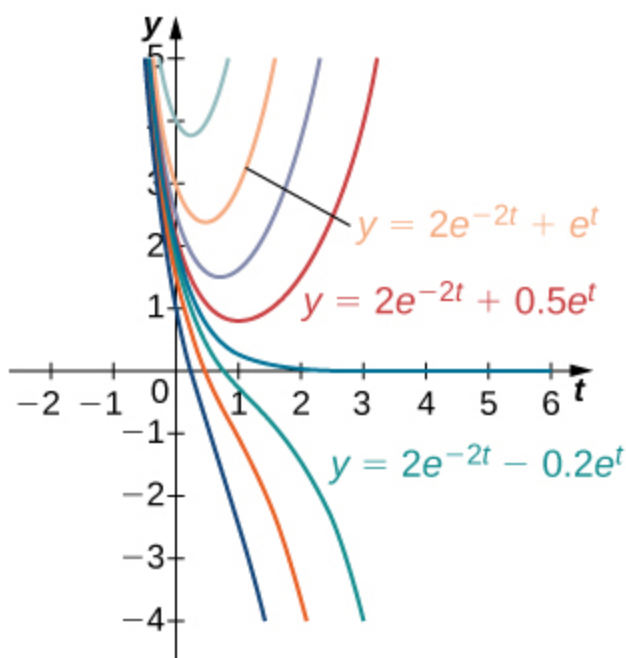
$$y' - 2y = 4\cos t - 8\sin t, \quad y(0) = 3.$$

Hint

First verify that y solves the differential equation. Then check the initial value.

In [\[link\]](#), the initial-value problem consisted of two parts. The first part was the differential equation $y' + 2y = 3e^x$, and the second part was the initial value $y(0) = 3$. These two equations together formed the initial-value problem.

The same is true in general. An initial-value problem will consists of two parts: the differential equation and the initial condition. The differential equation has a family of solutions, and the initial condition determines the value of C . The family of solutions to the differential equation in [\[link\]](#) is given by $y = 2e^{-2t} + Ce^t$. This family of solutions is shown in [\[link\]](#), with the particular solution $y = 2e^{-2t} + e^t$ labeled.



A family of solutions to the differential equation $y' + 2y = 3e^t$. The particular solution $y = 2e^{-2t} + e^t$ is labeled.

Example:

Exercise:

Problem:

Solving an Initial-value Problem

Solve the following initial-value problem:

Equation:

$$y' = 3e^x + x^2 - 4, \quad y(0) = 5.$$

Solution:

The first step in solving this initial-value problem is to find a general family of solutions. To do this, we find an antiderivative of both sides of the differential equation

Equation:

$$\int y' dx = \int (3e^x + x^2 - 4) dx,$$

namely,

Equation:

$$y + C_1 = 3e^x + \frac{1}{3}x^3 - 4x + C_2.$$

We are able to integrate both sides because the y term appears by itself. Notice that there are two integration constants: C_1 and C_2 . Solving [\[link\]](#) for y gives

Equation:

$$y = 3e^x + \frac{1}{3}x^3 - 4x + C_2 - C_1.$$

Because C_1 and C_2 are both constants, $C_2 - C_1$ is also a constant. We can therefore define $C = C_2 - C_1$, which leads to the equation

Equation:

$$y = 3e^x + \frac{1}{3}x^3 - 4x + C.$$

Next we determine the value of C . To do this, we substitute $x = 0$ and $y = 5$ into [\[link\]](#) and solve for C :

Equation:

$$\begin{aligned} 5 &= 3e^0 + \frac{1}{3}0^3 - 4(0) + C \\ 5 &= 3 + C \\ C &= 2. \end{aligned}$$

Now we substitute the value $C = 2$ into [\[link\]](#). The solution to the initial-value problem is $y = 3e^x + \frac{1}{3}x^3 - 4x + 2$.

Analysis

The difference between a general solution and a particular solution is that a general solution involves a family of functions, either explicitly or implicitly defined, of the independent variable. The initial value or values determine which particular solution in the family of solutions satisfies the desired conditions.

Note:

Exercise:

Problem: Solve the initial-value problem

Equation:

$$y' = x^2 - 4x + 3 - 6e^x, \quad y(0) = 8.$$

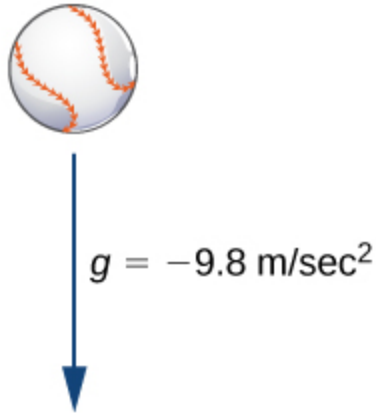
Solution:

$$y = \frac{1}{3}x^3 - 2x^2 + 3x - 6e^x + 14$$

Hint

First take the antiderivative of both sides of the differential equation. Then substitute $x = 0$ and $y = 8$ into the resulting equation and solve for C .

In physics and engineering applications, we often consider the forces acting upon an object, and use this information to understand the resulting motion that may occur. For example, if we start with an object at Earth's surface, the primary force acting upon that object is gravity. Physicists and engineers can use this information, along with Newton's second law of motion (in equation form $F = ma$, where F represents force, m represents mass, and a represents acceleration), to derive an equation that can be solved.



For a baseball falling in air, the only force acting on it is gravity (neglecting air resistance).

In [\[link\]](#) we assume that the only force acting on a baseball is the force of gravity. This assumption ignores air resistance. (The force due to air resistance is considered in a later discussion.) The acceleration due to gravity at Earth's surface, g , is approximately 9.8 m/s^2 . We introduce a frame of reference, where Earth's surface is at a height of 0 meters. Let $v(t)$ represent the velocity of the object in meters per second. If $v(t) > 0$, the ball is rising, and if $v(t) < 0$, the ball is falling ([\[link\]](#)).



Possible velocities for the
rising/falling baseball.

Our goal is to solve for the velocity $v(t)$ at any time t . To do this, we set up an initial-value problem. Suppose the mass of the ball is m , where m is measured in kilograms. We use Newton's second law, which states that the force acting on an object is equal to its mass times its acceleration ($F = ma$). Acceleration is the derivative of velocity, so $a(t) = v'(t)$. Therefore the force acting on the baseball is given by $F = m v'(t)$. However, this force must be equal to the force of gravity acting on the object, which (again using Newton's second law) is given by $F_g = -mg$, since this force acts in a downward direction. Therefore we obtain the equation $F = F_g$, which becomes $m v'(t) = -mg$. Dividing both sides of the equation by m gives the equation

Equation:

$$v'(t) = -g.$$

Notice that this differential equation remains the same regardless of the mass of the object.

We now need an initial value. Because we are solving for velocity, it makes sense in the context of the problem to assume that we know the **initial velocity**, or the velocity at time $t = 0$. This is denoted by $v(0) = v_0$.

Example:

Exercise:

Problem:

Velocity of a Moving Baseball

A baseball is thrown upward from a height of 3 meters above Earth's surface with an initial velocity of 10 m/s, and the only force acting on it is gravity. The ball has a mass of 0.15 kg at Earth's surface.

- Find the velocity $v(t)$ of the baseball at time t .
- What is its velocity after 2 seconds?

Solution:

- From the preceding discussion, the differential equation that applies in this situation is

Equation:

$$v'(t) = -g,$$

where $g = 9.8 \text{ m/s}^2$. The initial condition is $v(0) = v_0$, where $v_0 = 10 \text{ m/s}$. Therefore the initial-value problem is $v'(t) = -9.8 \text{ m/s}^2$, $v(0) = 10 \text{ m/s}$.

The first step in solving this initial-value problem is to take the antiderivative of both sides of the differential equation. This gives

Equation:

$$\begin{aligned}\int v'(t) dt &= \int -9.8 dt \\ v(t) &= -9.8t + C.\end{aligned}$$

The next step is to solve for C . To do this, substitute $t = 0$ and $v(0) = 10$:

Equation:

$$\begin{aligned}v(t) &= -9.8t + C \\ v(0) &= -9.8(0) + C \\ 10 &= C.\end{aligned}$$

Therefore $C = 10$ and the velocity function is given by $v(t) = -9.8t + 10$.

b. To find the velocity after 2 seconds, substitute $t = 2$ into $v(t)$.

Equation:

$$\begin{aligned}v(t) &= -9.8t + 10 \\ v(2) &= -9.8(2) + 10 \\ v(2) &= -9.6.\end{aligned}$$

The units of velocity are meters per second. Since the answer is negative, the object is falling at a speed of 9.6 m/s.

Note:

Exercise:

Problem:

Suppose a rock falls from rest from a height of 100 meters and the only force acting on it is gravity. Find an equation for the velocity $v(t)$ as a function of time, measured in meters per second.

Solution:

$$v(t) = -9.8t$$

Hint

What is the initial velocity of the rock? Use this with the differential equation in [\[link\]](#) to form an initial-value problem, then solve for $v(t)$.

A natural question to ask after solving this type of problem is how high the object will be above Earth's surface at a given point in time. Let $s(t)$ denote the height above Earth's surface of the object, measured in meters. Because velocity is the derivative of position (in this case height), this assumption gives the equation $s'(t) = v(t)$. An initial value is necessary; in this case the initial height of the object works well. Let the initial height be given by the equation $s(0) = s_0$. Together these assumptions give the initial-value problem

Equation:

$$s'(t) = v(t), \quad s(0) = s_0.$$

If the velocity function is known, then it is possible to solve for the position function as well.

Example:**Exercise:****Problem:**

Height of a Moving Baseball

A baseball is thrown upward from a height of 3 meters above Earth's surface with an initial velocity of 10 m/s, and the only force acting on it is gravity. The ball has a mass of 0.15 kilogram at Earth's surface.

- Find the position $s(t)$ of the baseball at time t .
- What is its height after 2 seconds?

Solution:

- We already know the velocity function for this problem is $v(t) = -9.8t + 10$. The initial height of the baseball is 3 meters, so $s_0 = 3$. Therefore the initial-value problem for this example is

To solve the initial-value problem, we first find the antiderivatives:

Equation:

$$\begin{aligned}\int s'(t) dt &= \int -9.8t + 10 dt \\ s(t) &= -4.9t^2 + 10t + C.\end{aligned}$$

Next we substitute $t = 0$ and solve for C :

Equation:

$$\begin{aligned}s(t) &= -4.9t^2 + 10t + C \\ s(0) &= -4.9(0)^2 + 10(0) + C \\ 3 &= C.\end{aligned}$$

Therefore the position function is $s(t) = -4.9t^2 + 10t + 3$.

- The height of the baseball after 2 s is given by $s(2)$:

Equation:

$$\begin{aligned}
 s(2) &= -4.9(2)^2 + 10(2) + 3 \\
 &= -4.9(4) + 23 \\
 &= 3.4.
 \end{aligned}$$

Therefore the baseball is 3.4 meters above Earth's surface after 2 seconds. It is worth noting that the mass of the ball cancelled out completely in the process of solving the problem.

Key Concepts

- A differential equation is an equation involving a function $y = f(x)$ and one or more of its derivatives. A solution is a function $y = f(x)$ that satisfies the differential equation when f and its derivatives are substituted into the equation.
- The order of a differential equation is the highest order of any derivative of the unknown function that appears in the equation.
- A differential equation coupled with an initial value is called an initial-value problem. To solve an initial-value problem, first find the general solution to the differential equation, then determine the value of the constant. Initial-value problems have many applications in science and engineering.

Determine the order of the following differential equations.

Exercise:

Problem: $y' + y = 3y^2$

Solution:

1

Exercise:

Problem: $(y')^2 = y' + 2y$

Exercise:

Problem: $y''' + y''y' = 3x^2$

Solution:

3

Exercise:

Problem: $y' = y'' + 3t^2$

Exercise:

Problem: $\frac{dy}{dt} = t$

Solution:

1

Exercise:

Problem: $\frac{dy}{dx} + \frac{d^2y}{dx^2} = 3x^4$

Exercise:

Problem: $\left(\frac{dy}{dt}\right)^2 + 8\frac{dy}{dt} + 3y = 4t$

Solution:

1

Verify that the following functions are solutions to the given differential equation.

Exercise:

Problem: $y = \frac{x^3}{3}$ solves $y' = x^2$

Exercise:

Problem: $y = 2e^{-x} + x - 1$ solves $y' = x - y$

Exercise:

Problem: $y = e^{3x} - \frac{e^x}{2}$ solves $y' = 3y + e^x$

Exercise:

Problem: $y = \frac{1}{1-x}$ solves $y' = y^2$

Exercise:

Problem: $y = e^{x^2/2}$ solves $y' = xy$

Exercise:

Problem: $y = 4 + \ln x$ solves $xy' = 1$

Exercise:

Problem: $y = 3 - x + x \ln x$ solves $y' = \ln x$

Exercise:

Problem: $y = 2e^x - x - 1$ solves $y' = y + x$

Exercise:

Problem: $y = e^x + \frac{\sin x}{2} - \frac{\cos x}{2}$ solves $y' = \cos x + y$

Exercise:

Problem: $y = \pi e^{-\cos x}$ solves $y' = y \sin x$

Verify the following general solutions and find the particular solution.

Exercise:**Problem:**

Find the particular solution to the differential equation $y' = 4x^2$ that passes through $(-3, -30)$, given that $y = C + \frac{4x^3}{3}$ is a general solution.

Exercise:**Problem:**

Find the particular solution to the differential equation $y' = 3x^3$ that passes through $(1, 4.75)$, given that $y = C + \frac{3x^4}{4}$ is a general solution.

Solution:

$$y = 4 + \frac{3x^4}{4}$$

Exercise:**Problem:**

Find the particular solution to the differential equation $y' = 3x^2y$ that passes through $(0, 12)$, given that $y = Ce^{x^3}$ is a general solution.

Exercise:**Problem:**

Find the particular solution to the differential equation $y' = 2xy$ that passes through $(0, \frac{1}{2})$, given that $y = Ce^{x^2}$ is a general solution.

Solution:

$$y = \frac{1}{2}e^{x^2}$$

Exercise:

Problem:

Find the particular solution to the differential equation $y' = (2xy)^2$ that passes through $(1, -\frac{1}{2})$, given that $y = -\frac{3}{C+4x^3}$ is a general solution.

Exercise:

Problem:

Find the particular solution to the differential equation $y'x^2 = y$ that passes through $(1, \frac{2}{e})$, given that $y = Ce^{-1/x}$ is a general solution.

Solution:

$$y = 2e^{-1/x}$$

Exercise:

Problem:

Find the particular solution to the differential equation $8\frac{dx}{dt} = -2\cos(2t) - \cos(4t)$ that passes through (π, π) , given that $x = C - \frac{1}{8}\sin(2t) - \frac{1}{32}\sin(4t)$ is a general solution.

Exercise:

Problem:

Find the particular solution to the differential equation $\frac{du}{dt} = \tan u$ that passes through $(1, \frac{\pi}{2})$, given that $u = \sin^{-1}(e^{C+t})$ is a general solution.

Solution:

$$u = \sin^{-1}(e^{-1+t})$$

Exercise:**Problem:**

Find the particular solution to the differential equation $\frac{dy}{dt} = e^{(t+y)}$ that passes through $(1, 0)$, given that $y = -\ln(C - e^t)$ is a general solution.

Exercise:**Problem:**

Find the particular solution to the differential equation $y'(1 - x^2) = 1 + y$ that passes through $(0, -2)$, given that $y = C \frac{\sqrt{x+1}}{\sqrt{1-x}} - 1$ is a general solution.

Solution:

$$y = -\frac{\sqrt{x+1}}{\sqrt{1-x}} - 1$$

For the following problems, find the general solution to the differential equation.

Exercise:

Problem: $y' = 3x + e^x$

Exercise:

Problem: $y' = \ln x + \tan x$

Solution:

$$y = C - x + x \ln x - \ln(\cos x)$$

Exercise:

Problem: $y' = \sin x e^{\cos x}$

Exercise:

Problem: $y' = 4^x$

Solution:

$$y = C + \frac{4^x}{\ln(4)}$$

Exercise:

Problem: $y' = \sin^{-1}(2x)$

Exercise:

Problem: $y' = 2t\sqrt{t^2 + 16}$

Solution:

$$y = \frac{2}{3}\sqrt{t^2 + 16}(t^2 + 16) + C$$

Exercise:

Problem: $x' = \coth t + \ln t + 3t^2$

Exercise:

Problem: $x' = t\sqrt{4 + t}$

Solution:

$$x = \frac{2}{15}\sqrt{4 + t}(3t^2 + 4t - 32) + C$$

Exercise:

Problem: $y' = y$

Exercise:

Problem: $y' = \frac{y}{x}$

Solution:

$$y = Cx$$

Solve the following initial-value problems starting from $y(t = 0) = 1$ and $y(t = 0) = -1$. Draw both solutions on the same graph.

Exercise:

Problem: $\frac{dy}{dt} = 2t$

Exercise:

Problem: $\frac{dy}{dt} = -t$

Solution:

$$y = 1 - \frac{t^2}{2}, y = -\frac{t^2}{2} - 1$$

Exercise:

Problem: $\frac{dy}{dt} = 2y$

Exercise:

Problem: $\frac{dy}{dt} = -y$

Solution:

$$y = e^{-t}, y = -e^{-t}$$

Exercise:

Problem: $\frac{dy}{dt} = 2$

Solve the following initial-value problems starting from $y_0 = 10$. At what time does y increase to 100 or drop to 1?

Exercise:

Problem: $\frac{dy}{dt} = 4t$

Solution:

$$y = 2(t^2 + 5), t = 3\sqrt{5}$$

Exercise:

Problem: $\frac{dy}{dt} = 4y$

Exercise:

Problem: $\frac{dy}{dt} = -2y$

Solution:

$$y = 10e^{-2t}, t = -\frac{1}{2} \ln\left(\frac{1}{10}\right)$$

Exercise:

Problem: $\frac{dy}{dt} = e^{4t}$

Exercise:

Problem: $\frac{dy}{dt} = e^{-4t}$

Solution:

$$y = \frac{1}{4}(41 - e^{-4t}), \text{ never}$$

Recall that a family of solutions includes solutions to a differential equation that differ by a constant. For the following problems, use your calculator to graph a family of solutions to the given differential equation. Use initial conditions from $y(t = 0) = -10$ to $y(t = 0) = 10$ increasing by 2. Is there some critical point where the behavior of the solution begins to change?

Exercise:

Problem: [T] $y' = y(x)$

Exercise:

Problem: [T] $xy' = y$

Solution:

Solution changes from increasing to decreasing at $y(0) = 0$

Exercise:

Problem: [T] $y' = t^3$

Exercise:

Problem:

[T] $y' = x + y$ (Hint: $y = Ce^x - x - 1$ is the general solution)

Solution:

Solution changes from increasing to decreasing at $y(0) = 0$

Exercise:

Problem: [T] $y' = x \ln x + \sin x$

Exercise:

Problem:

Find the general solution to describe the velocity of a ball of mass 1 lb that is thrown upward at a rate a ft/sec.

Solution:

$$v(t) = -32t + a$$

Exercise:**Problem:**

In the preceding problem, if the initial velocity of the ball thrown into the air is $a = 25$ ft/s, write the particular solution to the velocity of the ball. Solve to find the time when the ball hits the ground.

Exercise:**Problem:**

You throw two objects with differing masses m_1 and m_2 upward into the air with the same initial velocity a ft/s. What is the difference in their velocity after 1 second?

Solution:

0 ft/s

Exercise:**Problem:**

[T] You throw a ball of mass 1 kilogram upward with a velocity of $a = 25$ m/s on Mars, where the force of gravity is $g = -3.711$ m/s². Use your calculator to approximate how much longer the ball is in the air on Mars.

Exercise:

Problem:

[T] For the previous problem, use your calculator to approximate how much higher the ball went on Mars.

Solution:

4.86 meters

Exercise:**Problem:**

[T] A car on the freeway accelerates according to $a = 15 \cos(\pi t)$, where t is measured in hours. Set up and solve the differential equation to determine the velocity of the car if it has an initial speed of 51 mph. After 40 minutes of driving, what is the driver's velocity?

Exercise:**Problem:**

[T] For the car in the preceding problem, find the expression for the distance the car has traveled in time t , assuming an initial distance of 0. How long does it take the car to travel 100 miles? Round your answer to hours and minutes.

Solution:

$$x = 50t - \frac{15}{\pi^2} \cos(\pi t) + \frac{3}{\pi^2}, 2 \text{ hours } 1 \text{ minute}$$

Exercise:**Problem:**

[T] For the previous problem, find the total distance traveled in the first hour.

Exercise:

Problem:

Substitute $y = Be^{3t}$ into $y' - y = 8e^{3t}$ to find a particular solution.

Solution:

$$y = 4e^{3t}$$

Exercise:**Problem:**

Substitute $y = a \cos(2t) + b \sin(2t)$ into $y' + y = 4 \sin(2t)$ to find a particular solution.

Exercise:**Problem:**

Substitute $y = a + bt + ct^2$ into $y' + y = 1 + t^2$ to find a particular solution.

Solution:

$$y = 3 - 2t + t^2$$

Exercise:**Problem:**

Substitute $y = ae^t \cos t + be^t \sin t$ into $y' = 2e^t \cos t$ to find a particular solution.

Exercise:**Problem:**

Solve $y' = e^{kt}$ with the initial condition $y(0) = 0$ and solve $y' = 1$ with the same initial condition. As k approaches 0, what do you notice?

Solution:

$$y = \frac{1}{k}(e^{kt} - 1) \text{ and } y = x$$

Glossary

differential equation

an equation involving a function $y = y(x)$ and one or more of its derivatives

general solution (or family of solutions)

the entire set of solutions to a given differential equation

initial value(s)

a value or set of values that a solution of a differential equation satisfies for a fixed value of the independent variable

initial velocity

the velocity at time $t = 0$

initial-value problem

a differential equation together with an initial value or values

order of a differential equation

the highest order of any derivative of the unknown function that appears in the equation

particular solution

member of a family of solutions to a differential equation that satisfies a particular initial condition

solution to a differential equation

a function $y = f(x)$ that satisfies a given differential equation

Separable Equations

- Use separation of variables to solve a differential equation.
- Solve applications using separation of variables.

We now examine a solution technique for finding exact solutions to a class of differential equations known as separable differential equations. These equations are common in a wide variety of disciplines, including physics, chemistry, and engineering. We illustrate a few applications at the end of the section.

Separation of Variables

We start with a definition and some examples.

Note:

Definition

A **separable differential equation** is any equation that can be written in the form

Equation:

$$y' = f(x)g(y).$$

The term ‘separable’ refers to the fact that the right-hand side of the equation can be separated into a function of x times a function of y . Examples of separable differential equations include

Equation:

$$y' = (x^2 - 4)(3y + 2)$$

$$y' = 6x^2 + 4x$$

$$y' = \sec y + \tan y$$

$$y' = xy + 3x - 2y - 6.$$

The second equation is separable with $f(x) = 6x^2 + 4x$ and $g(y) = 1$, the third equation is separable with $f(x) = 1$ and $g(y) = \sec y + \tan y$, and the right-hand side of the fourth equation can be factored as $(x + 3)(y - 2)$, so it is separable as well. The third equation is also called an **autonomous differential equation** because the right-hand side of the equation is a function of y alone. If a differential equation is separable, then it is possible to solve the equation using the method of **separation of variables**.

Note:

Problem-Solving Strategy: Separation of Variables

1. Check for any values of y that make $g(y) = 0$. These correspond to constant solutions.
2. Rewrite the differential equation in the form $\frac{dy}{g(y)} = f(x)dx$.
3. Integrate both sides of the equation.
4. Solve the resulting equation for y if possible.
5. If an initial condition exists, substitute the appropriate values for x and y into the equation and solve for the constant.

Note that Step 4. states “Solve the resulting equation for y if possible.” It is not always possible to obtain y as an explicit function of x . Quite often we have to be satisfied with finding y as an implicit function of x .

Example:

Exercise:

Problem:
Using Separation of Variables

Find a general solution to the differential equation $y' = (x^2 - 4)(3y + 2)$ using the method of separation of variables.

Solution:

Follow the five-step method of separation of variables.

1. In this example, $f(x) = x^2 - 4$ and $g(y) = 3y + 2$. Setting $g(y) = 0$ gives $y = -\frac{2}{3}$ as a constant solution.
2. Rewrite the differential equation in the form

Equation:

$$\frac{dy}{3y + 2} = (x^2 - 4)dx.$$

3. Integrate both sides of the equation:

Equation:

$$\int \frac{dy}{3y + 2} = \int (x^2 - 4) dx.$$

Let $u = 3y + 2$. Then $du = 3\frac{dy}{dx}dx$, so the equation becomes

Equation:

$$\begin{aligned}\frac{1}{3} \int \frac{1}{u} du &= \frac{1}{3} x^3 - 4x + C \\ \frac{1}{3} \ln |u| &= \frac{1}{3} x^3 - 4x + C \\ \frac{1}{3} \ln |3y + 2| &= \frac{1}{3} x^3 - 4x + C.\end{aligned}$$

4. To solve this equation for y , first multiply both sides of the equation by 3.

Equation:

$$\ln |3y + 2| = x^3 - 12x + 3C$$

Now we use some logic in dealing with the constant C . Since C represents an arbitrary constant, $3C$ also represents an arbitrary constant. If we call the second arbitrary constant C_1 , the equation becomes

Equation:

$$\ln |3y + 2| = x^3 - 12x + C_1.$$

Now exponentiate both sides of the equation (i.e., make each side of the equation the exponent for the base e).

Equation:

$$\begin{aligned} e^{\ln|3y+2|} &= e^{x^3-12x+C_1} \\ |3y + 2| &= e^{C_1} e^{x^3-12x} \end{aligned}$$

Again define a new constant $C_2 = e^{C_1}$ (note that $C_2 > 0$):

Equation:

$$|3y + 2| = C_2 e^{x^3-12x}.$$

This corresponds to two separate equations: $3y + 2 = C_2 e^{x^3-12x}$ and $3y + 2 = -C_2 e^{x^3-12x}$.

The solution to either equation can be written in the form

$$y = \frac{-2 \pm C_2 e^{x^3-12x}}{3}.$$

Since $C_2 > 0$, it does not matter whether we use plus or minus, so the constant can actually have either sign. Furthermore, the subscript on the constant C is entirely arbitrary, and can be dropped. Therefore the solution can be written as

Equation:

$$y = \frac{-2 + Ce^{x^3-12x}}{3}.$$

5. No initial condition is imposed, so we are finished.

Note:

Exercise:

Problem:

Use the method of separation of variables to find a general solution to the differential equation $y' = 2xy + 3y - 4x - 6$.

Solution:

$$y = 2 + Ce^{x^2+3x}$$

Hint

First factor the right-hand side of the equation by grouping, then use the five-step strategy of separation of variables.

Example:

Exercise:

Problem:

Solving an Initial-Value Problem

Using the method of separation of variables, solve the initial-value problem

Equation:

$$y' = (2x + 3)(y^2 - 4), \quad y(0) = 1.$$

Solution:

Follow the five-step method of separation of variables.

1. In this example, $f(x) = 2x + 3$ and $g(y) = y^2 - 4$. Setting $g(y) = 0$ gives $y = \pm 2$ as constant solutions.
2. Divide both sides of the equation by $y^2 - 4$ and multiply by dx . This gives the equation

Equation:

$$\frac{dy}{y^2 - 4} = (2x + 3) dx.$$

3. Next integrate both sides:

Equation:

$$\int \frac{1}{y^2 - 4} dy = \int (2x + 3) dx.$$

To evaluate the left-hand side, use the method of partial fraction decomposition. This leads to the identity

Equation:

$$\frac{1}{y^2 - 4} = \frac{1}{4} \left(\frac{1}{y - 2} - \frac{1}{y + 2} \right).$$

Then [\[link\]](#) becomes

Equation:

$$\begin{aligned} \frac{1}{4} \int \left(\frac{1}{y - 2} - \frac{1}{y + 2} \right) dy &= \int (2x + 3) dx \\ \frac{1}{4} (\ln |y - 2| - \ln |y + 2|) &= x^2 + 3x + C. \end{aligned}$$

Multiplying both sides of this equation by 4 and replacing $4C$

with C_1 gives

Equation:

$$\begin{aligned}\ln |y - 2| - \ln |y + 2| &= 4x^2 + 12x + C_1 \\ \ln \left| \frac{y-2}{y+2} \right| &= 4x^2 + 12x + C_1.\end{aligned}$$

4. It is possible to solve this equation for y . First exponentiate both sides of the equation and define $C_2 = e^{C_1}$:

Equation:

$$\left| \frac{y - 2}{y + 2} \right| = C_2 e^{4x^2 + 12x}.$$

Next we can remove the absolute value and let C_2 be either positive or negative. Then multiply both sides by $y + 2$.

Equation:

$$\begin{aligned}y - 2 &= C_2 (y + 2) e^{4x^2 + 12x} \\ y - 2 &= C_2 y e^{4x^2 + 12x} + 2C_2 e^{4x^2 + 12x}.\end{aligned}$$

Now collect all terms involving y on one side of the equation, and solve for y :

Equation:

$$\begin{aligned}y - C_2 y e^{4x^2 + 12x} &= 2 + 2C_2 e^{4x^2 + 12x} \\ y(1 - C_2 e^{4x^2 + 12x}) &= 2 + 2C_2 e^{4x^2 + 12x} \\ y &= \frac{2 + 2C_2 e^{4x^2 + 12x}}{1 - C_2 e^{4x^2 + 12x}}.\end{aligned}$$

5. To determine the value of C_2 , substitute $x = 0$ and $y = -1$ into the general solution. Alternatively, we can put the same values into an earlier equation, namely the equation $\frac{y-2}{y+2} = C_2 e^{4x^2 + 12x}$. This is much easier to solve for C_2 :

Equation:

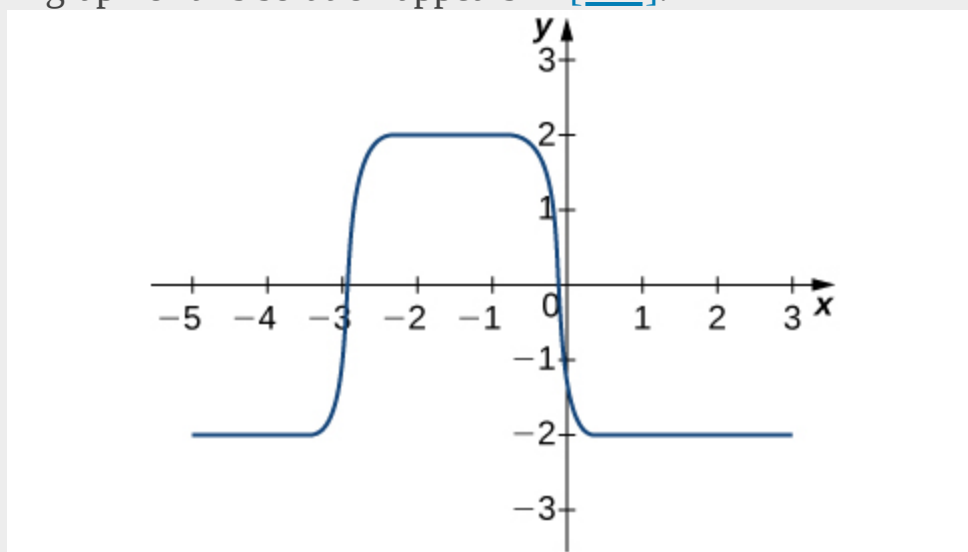
$$\begin{aligned}\frac{y-2}{y+2} &= C_2 e^{4x^2+12x} \\ \frac{-1-2}{-1+2} &= C_2 e^{4(0)^2+12(0)} \\ C_2 &= -3.\end{aligned}$$

Therefore the solution to the initial-value problem is

Equation:

$$y = \frac{2 - 6e^{4x^2+12x}}{1 + 3e^{4x^2+12x}}.$$

A graph of this solution appears in [\[link\]](#).



Graph of the solution to the initial-value problem
 $y' = (2x + 3)(y^2 - 4), \quad y(0) = -3.$

Note:

Exercise:

Problem: Find the solution to the initial-value problem

Equation:

$$6y' = (2x + 1)(y^2 - 2y - 8), \quad y(0) = -3$$

using the method of separation of variables.

Solution:

$$y = \frac{4 + 14e^{x^2 + x}}{1 - 7e^{x^2 + x}}$$

Hint

Follow the steps for separation of variables to solve the initial-value problem.

Applications of Separation of Variables

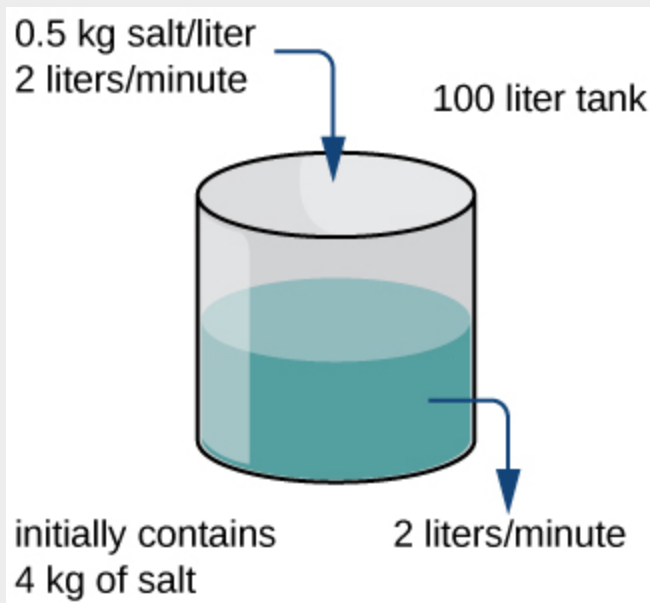
Many interesting problems can be described by separable equations. We illustrate two types of problems: solution concentrations and Newton's law of cooling.

Solution concentrations

Consider a tank being filled with a salt solution. We would like to determine the amount of salt present in the tank as a function of time. We can apply the process of separation of variables to solve this problem and similar problems involving solution concentrations.

Example:**Exercise:****Problem:****Determining Salt Concentration over Time**

A tank containing 100 L of a brine solution initially has 4 kg of salt dissolved in the solution. At time $t = 0$, another brine solution flows into the tank at a rate of 2 L/min. This brine solution contains a concentration of 0.5 kg/L of salt. At the same time, a stopcock is opened at the bottom of the tank, allowing the combined solution to flow out at a rate of 2 L/min, so that the level of liquid in the tank remains constant ([\[link\]](#)). Find the amount of salt in the tank as a function of time (measured in minutes), and find the limiting amount of salt in the tank, assuming that the solution in the tank is well mixed at all times.



A brine tank with an initial amount of salt solution accepts an input flow and delivers an output flow.

How does the amount of salt change with time?

Solution:

First we define a function $u(t)$ that represents the amount of salt in kilograms in the tank as a function of time. Then $\frac{du}{dt}$ represents the rate at which the amount of salt in the tank changes as a function of time. Also, $u(0)$ represents the amount of salt in the tank at time $t = 0$, which is 4 kilograms.

The general setup for the differential equation we will solve is of the form

Equation:

$$\frac{du}{dt} = \text{INFLOW RATE} - \text{OUTFLOW RATE}.$$

INFLOW RATE represents the rate at which salt enters the tank, and OUTFLOW RATE represents the rate at which salt leaves the tank. Because solution enters the tank at a rate of 2 L/min, and each liter of solution contains 0.5 kilogram of salt, every minute $2(0.5) = 1$ kilogram of salt enters the tank. Therefore INFLOW RATE = 1.

To calculate the rate at which salt leaves the tank, we need the concentration of salt in the tank at any point in time. Since the actual amount of salt varies over time, so does the concentration of salt. However, the volume of the solution remains fixed at 100 liters. The number of kilograms of salt in the tank at time t is equal to $u(t)$.

Thus, the concentration of salt is $\frac{u(t)}{100}$ kg/L, and the solution leaves the tank at a rate of 2 L/min. Therefore salt leaves the tank at a rate of $\frac{u(t)}{100} \cdot 2 = \frac{u(t)}{50}$ kg/min, and OUTFLOW RATE is equal to $\frac{u(t)}{50}$.

Therefore the differential equation becomes $\frac{du}{dt} = 1 - \frac{u}{50}$, and the initial condition is $u(0) = 4$. The initial-value problem to be solved is

Equation:

$$\frac{du}{dt} = 1 - \frac{u}{50}, \quad u(0) = 4.$$

The differential equation is a separable equation, so we can apply the five-step strategy for solution.

Step 1. Setting $1 - \frac{u}{50} = 0$ gives $u = 50$ as a constant solution. Since the initial amount of salt in the tank is 4 kilograms, this solution does not apply.

Step 2. Rewrite the equation as

Equation:

$$\frac{du}{dt} = \frac{50 - u}{50}.$$

Then multiply both sides by dt and divide both sides by $50 - u$:

Equation:

$$\frac{du}{50 - u} = \frac{dt}{50}.$$

Step 3. Integrate both sides:

Equation:

$$\begin{aligned} \int \frac{du}{50 - u} &= \int \frac{dt}{50} \\ -\ln |50 - u| &= \frac{t}{50} + C. \end{aligned}$$

Step 4. Solve for $u(t)$:

Equation:

$$\begin{aligned}\ln |50 - u| &= -\frac{t}{50} - C \\ e^{\ln |50 - u|} &= e^{-(t/50) - C} \\ |50 - u| &= C_1 e^{-t/50}.\end{aligned}$$

Eliminate the absolute value by allowing the constant to be either positive or negative:

Equation:

$$50 - u = C_1 e^{-t/50}.$$

Finally, solve for $u(t)$:

Equation:

$$u(t) = 50 - C_1 e^{-t/50}.$$

Step 5. Solve for C_1 :

Equation:

$$\begin{aligned}u(0) &= 50 - C_1 e^{-0/50} \\ 4 &= 50 - C_1 \\ C_1 &= 46.\end{aligned}$$

The solution to the initial value problem is $u(t) = 50 - 46e^{-t/50}$. To find the limiting amount of salt in the tank, take the limit as t approaches infinity:

Equation:

$$\begin{aligned}\lim_{t \rightarrow \infty} u(t) &= 50 - 46e^{-t/50} \\ &= 50 - 46(0) \\ &= 50.\end{aligned}$$

Note that this was the constant solution to the differential equation. If the initial amount of salt in the tank is 50 kilograms, then it remains constant. If it starts at less than 50 kilograms, then it approaches 50 kilograms over time.

Note:

Exercise:

Problem:

A tank contains 3 kilograms of salt dissolved in 75 liters of water. A salt solution of 0.4 kg salt/L is pumped into the tank at a rate of 6 L/min and is drained at the same rate. Solve for the salt concentration at time t . Assume the tank is well mixed at all times.

Solution:

Initial value problem:

$$\frac{du}{dt} = 2.4 - \frac{2u}{25}, \quad u(0) = 3$$

$$\text{Solution: } u(t) = 30 - 27e^{-t/50}$$

Hint

Follow the steps in [\[link\]](#) and determine an expression for INFLOW and OUTFLOW. Formulate an initial-value problem, and then solve it.

Newton's law of cooling

Newton's law of cooling states that the rate of change of an object's temperature is proportional to the difference between its own temperature and the ambient temperature (i.e., the temperature of its surroundings). If we let $T(t)$ represent the temperature of an object as a function of time,

then $\frac{dT}{dt}$ represents the rate at which that temperature changes. The temperature of the object's surroundings can be represented by T_s . Then Newton's law of cooling can be written in the form

Equation:

$$\frac{dT}{dt} = k (T(t) - T_s)$$

or simply

Equation:

$$\frac{dT}{dt} = k (T - T_s).$$

The temperature of the object at the beginning of any experiment is the initial value for the initial-value problem. We call this temperature T_0 . Therefore the initial-value problem that needs to be solved takes the form

Equation:

$$\frac{dT}{dt} = k (T - T_s), \quad T(0) = T_0,$$

where k is a constant that needs to be either given or determined in the context of the problem. We use these equations in [\[link\]](#).

Example:

Exercise:

Problem:

Waiting for a Pizza to Cool

A pizza is removed from the oven after baking thoroughly, and the temperature of the pizza when it comes out of the oven is 350°F . The temperature of the kitchen is 75°F , and after 5 minutes the temperature of the pizza is 340°F . We would like to wait until the temperature of the pizza reaches 300°F before cutting and serving it ([\[link\]](#)). How much longer will we have to wait?

Room temperature
is 75 degrees



Pizza temperature
is 350 degrees

From Newton's law of cooling, if the pizza cools 10°F in 5 minutes, how long before it cools to 300°F ?

Solution:

The ambient temperature (surrounding temperature) is 75°F , so $T_s = 75$. The temperature of the pizza when it comes out of the oven is 350°F , which is the initial temperature (i.e., initial value), so $T_0 = 350$. Therefore [\[link\]](#) becomes

Equation:

$$\frac{dT}{dt} = k(T - 75), \quad T(0) = 350.$$

To solve the differential equation, we use the five-step technique for solving separable equations.

1. Setting the right-hand side equal to zero gives $T = 75$ as a constant solution. Since the pizza starts at 350°F , this is not the solution we are seeking.
2. Rewrite the differential equation by multiplying both sides by dt and dividing both sides by $T - 75$:

Equation:

$$\frac{dT}{T - 75} = kdt.$$

3. Integrate both sides:

Equation:

$$\begin{aligned}\int \frac{dT}{T - 75} &= \int kdt \\ \ln |T - 75| &= kt + C.\end{aligned}$$

4. Solve for T by first exponentiating both sides:

Equation:

$$\begin{aligned}e^{\ln |T - 75|} &= e^{kt + C} \\ |T - 75| &= C_1 e^{kt} \\ T - 75 &= C_1 e^{kt} \\ T(t) &= 75 + C_1 e^{kt}.\end{aligned}$$

5. Solve for C_1 by using the initial condition $T(0) = 350$:

Equation:

$$\begin{aligned}T(t) &= 75 + C_1 e^{kt} \\ T(0) &= 75 + C_1 e^{k(0)} \\ 350 &= 75 + C_1 \\ C_1 &= 275.\end{aligned}$$

Therefore the solution to the initial-value problem is
Equation:

$$T(t) = 75 + 275e^{kt}.$$

To determine the value of k , we need to use the fact that after 5 minutes the temperature of the pizza is 340°F . Therefore $T(5) = 340$. Substituting this information into the solution to the initial-value problem, we have

Equation:

$$\begin{aligned}T(t) &= 75 + 275e^{kt} \\T(5) &= 340 = 75 + 275e^{5k} \\265 &= 275e^{5k} \\e^{5k} &= \frac{53}{55} \\\ln e^{5k} &= \ln\left(\frac{53}{55}\right) \\5k &= \ln\left(\frac{53}{55}\right) \\k &= \frac{1}{5} \ln\left(\frac{53}{55}\right) \approx -0.007408.\end{aligned}$$

So now we have $T(t) = 75 + 275e^{-0.007048t}$. When is the temperature 300°F ? Solving for t , we find

Equation:

$$\begin{aligned}
T(t) &= 75 + 275e^{-0.007048t} \\
300 &= 75 + 275e^{-0.007048t} \\
225 &= 275e^{-0.007048t} \\
e^{-0.007048t} &= \frac{9}{11} \\
\ln e^{-0.007048t} &= \ln \frac{9}{11} \\
-0.007048t &= \ln \frac{9}{11} \\
t &= -\frac{1}{0.007048} \ln \frac{9}{11} \approx 28.5.
\end{aligned}$$

Therefore we need to wait an additional 23.5 minutes (after the temperature of the pizza reached 340°F). That should be just enough time to finish this calculation.

Note:

Exercise:

Problem:

A cake is removed from the oven after baking thoroughly, and the temperature of the cake when it comes out of the oven is 450°F . The temperature of the kitchen is 70°F , and after 10 minutes the temperature of the cake is 430°F .

- Write the appropriate initial-value problem to describe this situation.
- Solve the initial-value problem for $T(t)$.
- How long will it take until the temperature of the cake is within 5°F of room temperature?

Solution:

- a. Initial-value problem

$$\frac{dT}{dt} = k(T - 70), \quad T(0) = 450$$
- b. $T(t) = 70 + 380e^{kt}$
- c. Approximately 114 minutes.

Hint

Determine the values of T_s and T_0 then use [\[link\]](#).

Key Concepts

- A separable differential equation is any equation that can be written in the form $y' = f(x)g(y)$.
- The method of separation of variables is used to find the general solution to a separable differential equation.

Key Equations

- **Separable differential equation**

$$y' = f(x)g(y)$$
- **Solution concentration**

$$\frac{du}{dt} = \text{INFLOW RATE} - \text{OUTFLOW RATE}$$
- **Newton's law of cooling**

$$\frac{dT}{dt} = k(T - T_s)$$

Solve the following initial-value problems with the initial condition $y_0 = 0$ and graph the solution.

Exercise:

Problem: $\frac{dy}{dt} = y + 1$

Solution:

$$y = e^t - 1$$

Exercise:

Problem: $\frac{dy}{dt} = y - 1$

Exercise:

Problem: $\frac{dy}{dt} = y + 1$

Solution:

$$y = 1 - e^{-t}$$

Exercise:

Problem: $\frac{dy}{dt} = -y - 1$

Find the general solution to the differential equation.

Exercise:

Problem: $x^2 y' = (x + 1)y$

Solution:

$$y = Cx e^{-1/x}$$

Exercise:

Problem: $y' = \tan(y)x$

Exercise:

Problem: $y' = 2xy^2$

Solution:

$$y = \frac{1}{C - x^2}$$

Exercise:

Problem: $\frac{dy}{dt} = y \cos(3t + 2)$

Exercise:

Problem: $2x \frac{dy}{dx} = y^2$

Solution:

$$y = -\frac{2}{C + \ln x}$$

Exercise:

Problem: $y' = e^y x^2$

Exercise:

Problem: $(1 + x)y' = (x + 2)(y - 1)$

Solution:

$$y = Ce^x (x + 1) + 1$$

Exercise:

Problem: $\frac{dx}{dt} = 3t^2 (x^2 + 4)$

Exercise:

Problem: $t \frac{dy}{dt} = \sqrt{1 - y^2}$

Solution:

$$y = \sin(\ln t + C)$$

Exercise:

Problem: $y' = e^x e^y$

Find the solution to the initial-value problem.

Exercise:

Problem: $y' = e^{y-x}, y(0) = 0$

Solution:

$$y = -\ln(e^{-x})$$

Exercise:

Problem: $y' = y^2(x + 1), y(0) = 2$

Exercise:

Problem: $\frac{dy}{dx} = y^3 x e^{x^2}, y(0) = 1$

Solution:

$$y = \frac{1}{\sqrt{2 - e^{x^2}}}$$

Exercise:

Problem: $\frac{dy}{dt} = y^2 e^x \sin(3x), y(0) = 1$

Exercise:

Problem: $y' = \frac{x}{\operatorname{sech}^2 y}, y(0) = 0$

Solution:

$$y = \tanh^{-1} \left(\frac{x^2}{2} \right)$$

Exercise:

Problem: $y' = 2xy(1 + 2y), y(0) = -1$

Exercise:

Problem: $\frac{dx}{dt} = \ln(t)\sqrt{1 - x^2}, x(1) = 0$

Solution:

$$x = \sin(1 - t + t \ln t)$$

Exercise:

Problem: $y' = 3x^2(y^2 + 4), y(0) = 0$

Exercise:

Problem: $y' = e^y 5^x, y(0) = \ln(\ln(5))$

Solution:

$$y = \ln(\ln(5)) - \ln(2 - 5^x)$$

Exercise:

Problem: $y' = -2x \tan(y), y(0) = \frac{\pi}{2}$

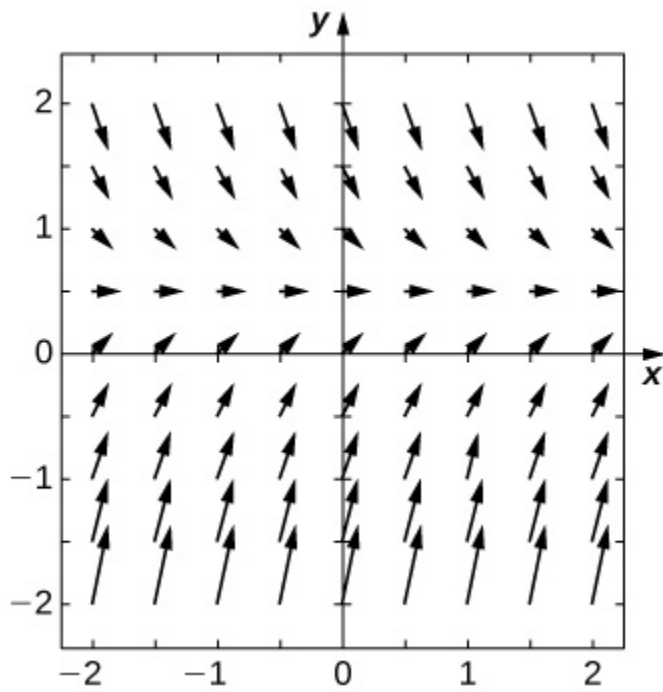
For the following problems, use a software program or your calculator to generate the directional fields. Solve explicitly and draw solution curves for several initial conditions. Are there some critical initial conditions that change the behavior of the solution?

Exercise:

Problem: [T] $y' = 1 - 2y$

Solution:

$$y = Ce^{-2x} + \frac{1}{2}$$



Exercise:

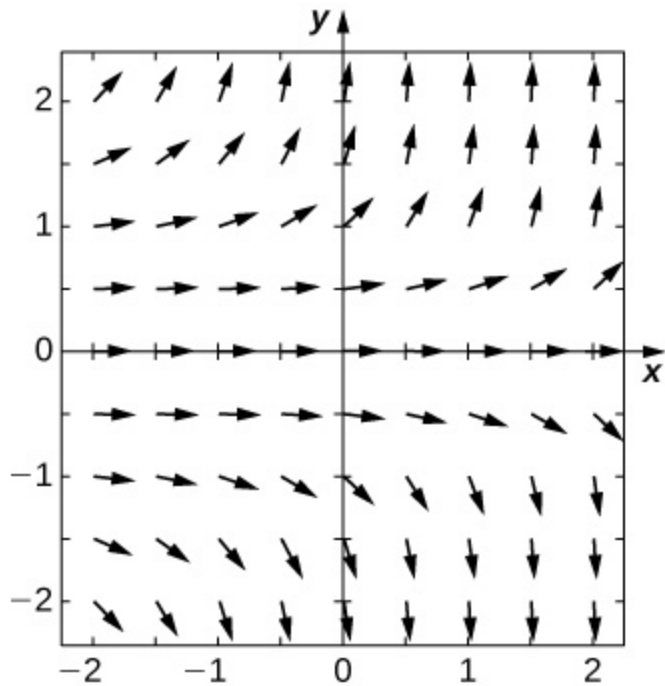
Problem: [T] $y' = y^2 x^3$

Exercise:

Problem: [T] $y' = y^3 e^x$

Solution:

$$y = \frac{1}{\sqrt{2}\sqrt{C - e^x}}$$



Exercise:

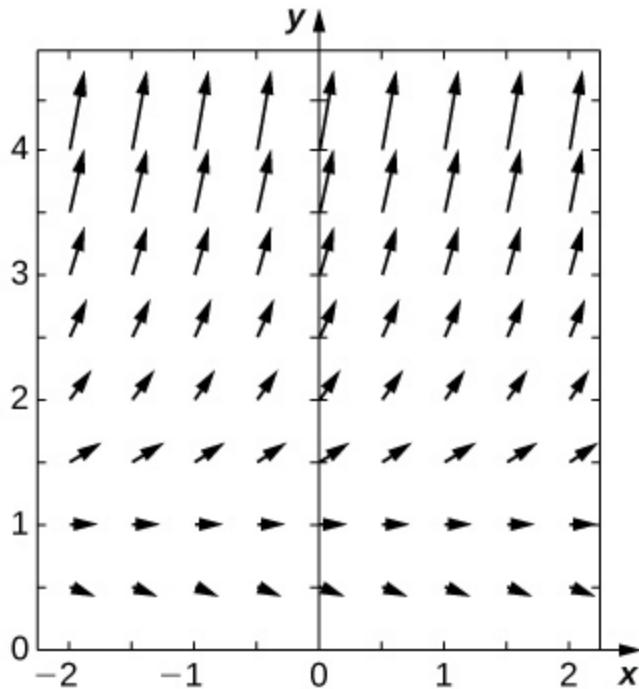
Problem: [T] $y' = e^y$

Exercise:

Problem: [T] $y' = y \ln(x)$

Solution:

$$y = Ce^{-x} x^x$$



Exercise:

Problem:

Most drugs in the bloodstream decay according to the equation $y' = -cy$, where y is the concentration of the drug in the bloodstream. If the half-life of a drug is 2 hours, what fraction of the initial dose remains after 6 hours?

Exercise:

Problem:

A drug is administered intravenously to a patient at a rate r mg/h and is cleared from the body at a rate proportional to the amount of drug still present in the body, d . Set up and solve the differential equation, assuming there is no drug initially present in the body.

Solution:

$$y = \frac{r}{d} (1 - e^{-dt})$$

Exercise:

Problem:

[T] How often should a drug be taken if its dose is 3 mg, it is cleared at a rate $c = 0.1$ mg/h, and 1 mg is required to be in the bloodstream at all times?

Exercise:**Problem:**

A tank contains 1 kilogram of salt dissolved in 100 liters of water. A salt solution of 0.1 kg salt/L is pumped into the tank at a rate of 2 L/min and is drained at the same rate. Solve for the salt concentration at time t . Assume the tank is well mixed.

Solution:

$$y(t) = 10 - 9e^{-t/50}$$

Exercise:**Problem:**

A tank containing 10 kilograms of salt dissolved in 1000 liters of water has two salt solutions pumped in. The first solution of 0.2 kg salt/L is pumped in at a rate of 20 L/min and the second solution of 0.05 kg salt/L is pumped in at a rate of 5 L/min. The tank drains at 25 L/min. Assume the tank is well mixed. Solve for the salt concentration at time t .

Exercise:**Problem:**

[T] For the preceding problem, find how much salt is in the tank 1 hour after the process begins.

Solution:

134.3 kilograms

Exercise:**Problem:**

Torricelli's law states that for a water tank with a hole in the bottom that has a cross-section of A and with a height of water h above the bottom of the tank, the rate of change of volume of water flowing from the tank is proportional to the square root of the height of water, according to $\frac{dV}{dt} = -A\sqrt{2gh}$, where g is the acceleration due to gravity. Note that $\frac{dV}{dt} = A\frac{dh}{dt}$. Solve the resulting initial-value problem for the height of the water, assuming a tank with a hole of radius 2 ft. The initial height of water is 100 ft.

Exercise:**Problem:**

For the preceding problem, determine how long it takes the tank to drain.

Solution:

720 seconds

For the following problems, use Newton's law of cooling.

Exercise:**Problem:**

The liquid base of an ice cream has an initial temperature of 200°F before it is placed in a freezer with a constant temperature of 0°F . After 1 hour, the temperature of the ice-cream base has decreased to 140°F . Formulate and solve the initial-value problem to determine the temperature of the ice cream.

Exercise:

Problem:

[T] The liquid base of an ice cream has an initial temperature of 210°F before it is placed in a freezer with a constant temperature of 20°F . After 2 hours, the temperature of the ice-cream base has decreased to 170°F . At what time will the ice cream be ready to eat? (Assume 30°F is the optimal eating temperature.)

Solution:

24 hours 57 minutes

Exercise:**Problem:**

[T] You are organizing an ice cream social. The outside temperature is 80°F and the ice cream is at 10°F . After 10 minutes, the ice cream temperature has risen by 10°F . How much longer can you wait before the ice cream melts at 40°F ?

Exercise:**Problem:**

You have a cup of coffee at temperature 70°C and the ambient temperature in the room is 20°C . Assuming a cooling rate k of 0.125, write and solve the differential equation to describe the temperature of the coffee with respect to time.

Solution:

$$T(t) = 20 + 50e^{-0.125t}$$

Exercise:**Problem:**

[T] You have a cup of coffee at temperature 70°C that you put outside, where the ambient temperature is 0°C . After 5 minutes, how much colder is the coffee?

Exercise:**Problem:**

You have a cup of coffee at temperature 70°C and you immediately pour in 1 part milk to 5 parts coffee. The milk is initially at temperature 1°C . Write and solve the differential equation that governs the temperature of this coffee.

Solution:

$$T(t) = 20 + 38.5e^{-0.125t}$$

Exercise:**Problem:**

You have a cup of coffee at temperature 70°C , which you let cool 10 minutes before you pour in the same amount of milk at 1°C as in the preceding problem. How does the temperature compare to the previous cup after 10 minutes?

Exercise:**Problem:**

Solve the generic problem $y' = ay + b$ with initial condition $y(0) = c$.

Solution:

$$y = \left(c + \frac{b}{a}\right)e^{ax} - \frac{b}{a}$$

Exercise:**Problem:**

Prove the basic continual compounded interest equation. Assuming an initial deposit of P_0 and an interest rate of r , set up and solve an equation for continually compounded interest.

Exercise:

Problem:

Assume an initial nutrient amount of I kilograms in a tank with L liters. Assume a concentration of c kg/L being pumped in at a rate of r L/min. The tank is well mixed and is drained at a rate of r L/min. Find the equation describing the amount of nutrient in the tank.

Solution:

$$y(t) = cL + (I - cL)e^{-rt/L}$$

Exercise:**Problem:**

Leaves accumulate on the forest floor at a rate of 2 g/cm²/yr and also decompose at a rate of 90% per year. Write a differential equation governing the number of grams of leaf litter per square centimeter of forest floor, assuming at time 0 there is no leaf litter on the ground. Does this amount approach a steady value? What is that value?

Exercise:**Problem:**

Leaves accumulate on the forest floor at a rate of 4 g/cm²/yr. These leaves decompose at a rate of 10% per year. Write a differential equation governing the number of grams of leaf litter per square centimeter of forest floor. Does this amount approach a steady value? What is that value?

Solution:

$$y = 40(1 - e^{-0.1t}), 40 \text{ g/cm}^2$$

Glossary

autonomous differential equation

an equation in which the right-hand side is a function of y alone

separable differential equation

any equation that can be written in the form $y' = f(x)g(y)$

separation of variables

a method used to solve a separable differential equation

First-order Linear Equations

- Write a first-order linear differential equation in standard form.
- Find an integrating factor and use it to solve a first-order linear differential equation.
- Solve applied problems involving first-order linear differential equations.

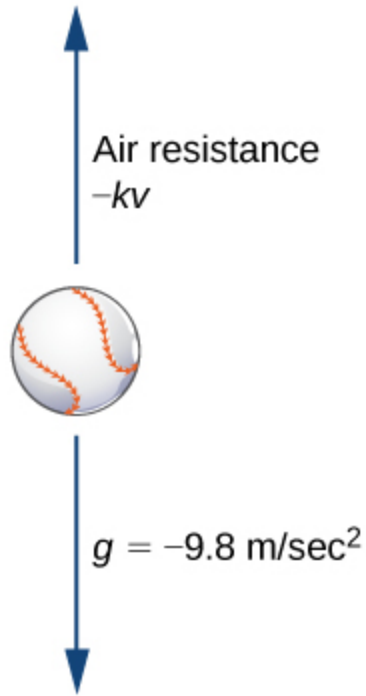
Earlier, we studied an application of a first-order differential equation that involved solving for the velocity of an object. In particular, if a ball is thrown upward with an initial velocity of v_0 ft/s, then an initial-value problem that describes the velocity of the ball after t seconds is given by

Equation:

$$\frac{dv}{dt} = -32, \quad v(0) = v_0.$$

This model assumes that the only force acting on the ball is gravity. Now we add to the problem by allowing for the possibility of air resistance acting on the ball.

Air resistance always acts in the direction opposite to motion. Therefore if an object is rising, air resistance acts in a downward direction. If the object is falling, air resistance acts in an upward direction ([\[link\]](#)). There is no exact relationship between the velocity of an object and the air resistance acting on it. For very small objects, air resistance is proportional to velocity; that is, the force due to air resistance is numerically equal to some constant k times v . For larger (e.g., baseball-sized) objects, depending on the shape, air resistance can be approximately proportional to the square of the velocity. In fact, air resistance may be proportional to $v^{1.5}$, or $v^{0.9}$, or some other power of v .



Forces acting on a moving baseball: gravity acts in a downward direction and air resistance acts in a direction opposite to the direction of motion.

We will work with the linear approximation for air resistance. If we assume $k > 0$, then the expression for the force F_A due to air resistance is given by $F_A = -kv$. Therefore the sum of the forces acting on the object is equal to the sum of the gravitational force and the force due to air resistance. This, in turn, is equal to the mass of the object multiplied by its acceleration at time t (Newton's second law). This gives us the differential equation

Equation:

$$m \frac{dv}{dt} = -kv - mg.$$

Finally, we impose an initial condition $v(0) = v_0$, where v_0 is the initial velocity measured in meters per second. This makes $g = 9.8 \text{ m/s}^2$. The initial-value problem becomes

Equation:

$$m \frac{dv}{dt} = -kv - mg, \quad v(0) = v_0.$$

The differential equation in this initial-value problem is an example of a first-order linear differential equation. (Recall that a differential equation is first-order if the highest-order derivative that appears in the equation is 1.) In this section, we study first-order linear equations and examine a method for finding a general solution to these types of equations, as well as solving initial-value problems involving them.

Note:

Definition

A first-order differential equation is **linear** if it can be written in the form

Equation:

$$a(x)y' + b(x)y = c(x),$$

where $a(x)$, $b(x)$, and $c(x)$ are arbitrary functions of x .

Remember that the unknown function y depends on the variable x ; that is, x is the independent variable and y is the dependent variable. Some examples of first-order linear differential equations are

Equation:

$$\begin{aligned} (3x^2 - 4)y' + (x - 3)y &= \sin x \\ (\sin x)y' - (\cos x)y &= \cot x \\ 4xy' + (3 \ln x)y &= x^3 - 4x. \end{aligned}$$

Examples of first-order nonlinear differential equations include

Equation:

$$\begin{aligned}(y')^4 - (y')^3 &= (3x - 2)(y + 4) \\ 4y' + 3y^3 &= 4x - 5 \\ (y')^2 &= \sin y + \cos x.\end{aligned}$$

These equations are nonlinear because of terms like $(y')^4$, y^3 , etc. Due to these terms, it is impossible to put these equations into the same form as [\[link\]](#).

Standard Form

Consider the differential equation

Equation:

$$(3x^2 - 4)y' + (x - 3)y = \sin x.$$

Our main goal in this section is to derive a solution method for equations of this form. It is useful to have the coefficient of y' be equal to 1. To make this happen, we divide both sides by $3x^2 - 4$.

Equation:

$$y' + \left(\frac{x - 3}{3x^2 - 4} \right) y = \frac{\sin x}{3x^2 - 4}$$

This is called the **standard form** of the differential equation. We will use it later when finding the solution to a general first-order linear differential equation. Returning to [\[link\]](#), we can divide both sides of the equation by $a(x)$. This leads to the equation

Equation:

$$y' + \frac{b(x)}{a(x)}y = \frac{c(x)}{a(x)}.$$

Now define $p(x) = \frac{b(x)}{a(x)}$ and $q(x) = \frac{c(x)}{a(x)}$. Then [\[link\]](#) becomes

Equation:

$$y' + p(x)y = q(x).$$

We can write any first-order linear differential equation in this form, and this is referred to as the standard form for a first-order linear differential equation.

Example:

Exercise:

Problem:

Writing First-Order Linear Equations in Standard Form

Put each of the following first-order linear differential equations into standard form. Identify $p(x)$ and $q(x)$ for each equation.

a. $y' = 3x - 4y$

b. $\frac{3xy'}{4y-3} = 2$ (here $x \neq 0$)

c. $y = 3y' - 4x^2 + 5$

Solution:

a. Add $4y$ to both sides:

Equation:

$$y' + 4y = 3x.$$

In this equation, $p(x) = 4$ and $q(x) = 3x$.

- b. Multiply both sides by $4y - 3$, then subtract $8y$ from each side:

Equation:

$$\begin{aligned}\frac{3xy'}{4y-3} &= 2 \\ 3xy' &= 2(4y-3) \\ 3xy' &= 8y-6 \\ 3xy'-8y &= -6.\end{aligned}$$

Finally, divide both sides by $3x$ to make the coefficient of y' equal to 1:

Equation:

$$y' - \frac{8}{3x}y = -\frac{2}{3x}.$$

This is allowable because in the original statement of this problem we assumed that $x \neq 0$. (If $x = 0$ then the original equation becomes $0 = 2$, which is clearly a false statement.)

In this equation, $p(x) = -\frac{8}{3x}$ and $q(x) = -\frac{2}{3x}$.

- c. Subtract y from each side and add $4x^2 - 5$:

Equation:

$$3y' - y = 4x^2 - 5.$$

Next divide both sides by 3:

Equation:

$$y' - \frac{1}{3}y = \frac{4}{3}x^2 - \frac{5}{3}.$$

In this equation, $p(x) = -\frac{1}{3}$ and $q(x) = \frac{4}{3}x^2 - \frac{5}{3}$.

Note:**Exercise:****Problem:**

Put the equation $\frac{(x+3)y'}{2x-3y-4} = 5$ into standard form and identify $p(x)$ and $q(x)$.

Solution:

$$y' + \frac{15}{x+3}y = \frac{10x-20}{x+3}; p(x) = \frac{15}{x+3} \text{ and } q(x) = \frac{10x-20}{x+3}$$

Hint

Multiply both sides by the common denominator, then collect all terms involving y on one side.

Integrating Factors

We now develop a solution technique for any first-order linear differential equation. We start with the standard form of a first-order linear differential equation:

Equation:

$$y' + p(x)y = q(x).$$

The first term on the left-hand side of [\[link\]](#) is the derivative of the unknown function, and the second term is the product of a known function with the unknown function. This is somewhat reminiscent of the product rule from the [Differentiation Rules](#) section. If we multiply [\[link\]](#) by a yet-to-be-determined function $\mu(x)$, then the equation becomes

Equation:

$$\mu(x)y' + \mu(x)p(x)y = \mu(x)q(x).$$

The left-hand side [\[link\]](#) can be matched perfectly to the product rule:

Equation:

$$\frac{d}{dx}[f(x)g(x)] = f'(x)g(x) + f(x)g'(x).$$

Matching term by term gives $y = f(x)$, $g(x) = \mu(x)$, and $g'(x) = \mu(x)p(x)$. Taking the derivative of $g(x) = \mu(x)$ and setting it equal to the right-hand side of $g'(x) = \mu(x)p(x)$ leads to

Equation:

$$\mu'(x) = \mu(x)p(x).$$

This is a first-order, separable differential equation for $\mu(x)$. We know $p(x)$ because it appears in the differential equation we are solving.

Separating variables and integrating yields

Equation:

$$\begin{aligned}\frac{\mu'(x)}{\mu(x)} &= p(x) \\ \int \frac{\mu'(x)}{\mu(x)} dx &= \int p(x) dx \\ \ln |\mu(x)| &= \int p(x) dx + C \\ e^{\ln |\mu(x)|} &= e^{\int p(x) dx + C} \\ |\mu(x)| &= C_1 e^{\int p(x) dx} \\ \mu(x) &= C_2 e^{\int p(x) dx}.\end{aligned}$$

Here C_2 can be an arbitrary (positive or negative) constant. This leads to a general method for solving a first-order linear differential equation. We first multiply both sides of [\[link\]](#) by the **integrating factor** $\mu(x)$. This gives

Equation:

$$\mu(x)y' + \mu(x)p(x)y = \mu(x)q(x).$$

The left-hand side of [\[link\]](#) can be rewritten as $\frac{d}{dx}(\mu(x)y)$.

Equation:

$$\frac{d}{dx}(\mu(x)y) = \mu(x)q(x).$$

Next integrate both sides of [\[link\]](#) with respect to x .

Equation:

$$\begin{aligned}\int \frac{d}{dx}(\mu(x)y) dx &= \int \mu(x)q(x) dx \\ \mu(x)y &= \int \mu(x)q(x) dx.\end{aligned}$$

Divide both sides of [\[link\]](#) by $\mu(x)$:

Equation:

$$y = \frac{1}{\mu(x)} \left[\int \mu(x)q(x) dx + C \right].$$

Since $\mu(x)$ was previously calculated, we are now finished. An important note about the integrating constant C : It may seem that we are inconsistent in the usage of the integrating constant. However, the integral involving $p(x)$ is necessary in order to find an integrating factor for [\[link\]](#). Only one integrating factor is needed in order to solve the equation; therefore, it is safe to assign a value for C for this integral. We chose $C = 0$. When calculating the integral inside the brackets in [\[link\]](#), it is necessary to keep

our options open for the value of the integrating constant, because our goal is to find a general family of solutions to [\[link\]](#). This integrating factor guarantees just that.

Note:

Problem-Solving Strategy: Solving a First-order Linear Differential Equation

1. Put the equation into standard form and identify $p(x)$ and $q(x)$.
2. Calculate the integrating factor $\mu(x) = e^{\int p(x) dx}$.
3. Multiply both sides of the differential equation by $\mu(x)$.
4. Integrate both sides of the equation obtained in step 3, and divide both sides by $\mu(x)$.
5. If there is an initial condition, determine the value of C .

Example:

Exercise:

Problem:

Solving a First-order Linear Equation

Find a general solution for the differential equation $xy' + 3y = 4x^2 - 3x$. Assume $x > 0$.

Solution:

1. To put this differential equation into standard form, divide both sides by x :

Equation:

$$y' + \frac{3}{x}y = 4x - 3.$$

Therefore $p(x) = \frac{3}{x}$ and $q(x) = 4x - 3$.

2. The integrating factor is $\mu(x) = e^{\int (3/x) dx} = e^{3 \ln x} = x^3$.

3. Multiplying both sides of the differential equation by $\mu(x)$ gives us

Equation:

$$x^3 y' + x^3 \left(\frac{3}{x}\right) y = x^3 (4x - 3)$$

$$x^3 y' + 3x^2 y = 4x^4 - 3x^3$$

$$\frac{d}{dx} (x^3 y) = 4x^4 - 3x^3.$$

4. Integrate both sides of the equation.

Equation:

$$\int \frac{d}{dx} (x^3 y) dx = \int 4x^4 - 3x^3 dx$$

$$x^3 y = \frac{4x^5}{5} - \frac{3x^4}{4} + C$$

$$y = \frac{4x^2}{5} - \frac{3x}{4} + Cx^{-3}.$$

5. There is no initial value, so the problem is complete.

Analysis

You may have noticed the condition that was imposed on the differential equation; namely, $x > 0$. For any nonzero value of C , the general solution is not defined at $x = 0$. Furthermore, when $x < 0$, the integrating factor

changes. The integrating factor is given by [\[link\]](#) as $f(x) = e^{\int p(x) dx}$. For this $p(x)$ we get

Equation:

$$e^{\int p(x) dx} = e^{\int (3/x) dx} = e^{3 \ln |x|} = |x|^3,$$

since $x < 0$. The behavior of the general solution changes at $x = 0$ largely due to the fact that $p(x)$ is not defined there.

Note:

Exercise:

Problem:

Find the general solution to the differential equation $(x - 2)y' + y = 3x^2 + 2x$. Assume $x > 2$.

Solution:

$$y = \frac{x^3 + x^2 + C}{x - 2}$$

Hint

Use the method outlined in the problem-solving strategy for first-order linear differential equations.

Now we use the same strategy to find the solution to an initial-value problem.

Example:

Exercise:

Problem:

A First-order Linear Initial-Value Problem

Solve the initial-value problem

Equation:

$$y' + 3y = 2x - 1, \quad y(0) = 3.$$

Solution:

1. This differential equation is already in standard form with $p(x) = 3$ and $q(x) = 2x - 1$.
2. The integrating factor is $\mu(x) = e^{\int 3dx} = e^{3x}$.
3. Multiplying both sides of the differential equation by $\mu(x)$ gives

Equation:

$$\begin{aligned}e^{3x}y' + 3e^{3x}y &= (2x - 1)e^{3x} \\ \frac{d}{dx}[ye^{3x}] &= (2x - 1)e^{3x}.\end{aligned}$$

Integrate both sides of the equation:

Equation:

$$\begin{aligned}\int \frac{d}{dx}[ye^{3x}]dx &= \int (2x - 1)e^{3x}dx \\ ye^{3x} &= \frac{e^{3x}}{3}(2x - 1) - \int \frac{2}{3}e^{3x}dx \\ ye^{3x} &= \frac{e^{3x}(2x-1)}{3} - \frac{2e^{3x}}{9} + C \\ y &= \frac{2x-1}{3} - \frac{2}{9} + Ce^{-3x} \\ y &= \frac{2x}{3} - \frac{5}{9} + Ce^{-3x}.\end{aligned}$$

4. Now substitute $x = 0$ and $y = 3$ into the general solution and solve for C :

Equation:

$$\begin{aligned}y &= \frac{2}{3}x - \frac{5}{9} + Ce^{-3x} \\ 3 &= \frac{2}{3}(0) - \frac{5}{9} + Ce^{-3(0)} \\ 3 &= -\frac{5}{9} + C \\ C &= \frac{32}{9}.\end{aligned}$$

Therefore the solution to the initial-value problem is
Equation:

$$y = \frac{2}{3}x - \frac{5}{9} + \frac{32}{9}e^{-3x}.$$

Note:

Exercise:

Problem:

Solve the initial-value problem $y' - 2y = 4x + 3$ $y(0) = -2$.

Solution:

$$y = -2x - 4 + 2e^{2x}$$

Applications of First-order Linear Differential Equations

We look at two different applications of first-order linear differential equations. The first involves air resistance as it relates to objects that are rising or falling; the second involves an electrical circuit. Other applications are numerous, but most are solved in a similar fashion.

Free fall with air resistance

We discussed air resistance at the beginning of this section. The next example shows how to apply this concept for a ball in vertical motion.

Other factors can affect the force of air resistance, such as the size and shape of the object, but we ignore them here.

Example:

Exercise:

Problem:

A Ball with Air Resistance

A racquetball is hit straight upward with an initial velocity of 2 m/s. The mass of a racquetball is approximately 0.0427 kg. Air resistance acts on the ball with a force numerically equal to $0.5v$, where v represents the velocity of the ball at time t .

- Find the velocity of the ball as a function of time.
- How long does it take for the ball to reach its maximum height?
- If the ball is hit from an initial height of 1 meter, how high will it reach?

Solution:

- The mass $m = 0.0427$ kg, $k = 0.5$, and $g = 9.8$ m/s². The initial velocity is $v_0 = 2$ m/s. Therefore the initial-value problem is

Equation:

$$0.0427 \frac{dv}{dt} = -0.5v - 0.0427(9.8), \quad v_0 = 2.$$

Dividing the differential equation by 0.0427 gives

Equation:

$$\frac{dv}{dt} = -11.7096v - 9.8, \quad v_0 = 2.$$

The differential equation is linear. Using the problem-solving strategy for linear differential equations:

Step 1. Rewrite the differential equation as

$$\frac{dv}{dt} + 11.7096v = -9.8. \text{ This gives } p(t) = 11.7096 \text{ and } q(t) = -9.8$$

Step 2. The integrating factor is $\mu(t) = e^{\int 11.7096 dt} = e^{11.7096t}$.

Step 3. Multiply the differential equation by $\mu(t)$:

Equation:

$$\begin{aligned} e^{11.7096t} \frac{dv}{dt} + 11.7096ve^{11.7096t} &= -9.8e^{11.7096t} \\ \frac{d}{dt} [ve^{11.7096t}] &= -9.8e^{11.7096t}. \end{aligned}$$

Step 4. Integrate both sides:

Equation:

$$\begin{aligned} \int \frac{d}{dt} [ve^{11.7096t}] dt &= \int -9.8e^{11.7096t} dt \\ ve^{11.7096t} &= \frac{-9.8}{11.7096} e^{11.7096t} + C \\ v(t) &= -0.8369 + Ce^{-11.7096t}. \end{aligned}$$

Step 5. Solve for C using the initial condition $v_0 = v(0) = 2$:

Equation:

$$\begin{aligned} v(t) &= -0.8369 + Ce^{-11.7096t} \\ v(0) &= -0.8369 + Ce^{-11.7096(0)} \\ 2 &= -0.8369 + C \\ C &= 2.8369. \end{aligned}$$

Therefore the solution to the initial-value problem is

$$v(t) = 2.8369e^{-11.7096t} - 0.8369.$$

- b. The ball reaches its maximum height when the velocity is equal to zero. The reason is that when the velocity is positive, it is rising, and when it is negative, it is falling. Therefore when it is zero, it is neither rising nor falling, and is at its maximum height:

Equation:

$$\begin{aligned}
 2.8369e^{-11.7096t} - 0.8369 &= 0 \\
 2.8369e^{-11.7096t} &= 0.8369 \\
 e^{-11.7096t} &= \frac{0.8369}{2.8369} \approx 0.295 \\
 \ln e^{-11.7096t} &= \ln 0.295 \approx -1.221 \\
 -11.7096t &= -1.221 \\
 t &\approx 0.104.
 \end{aligned}$$

Therefore it takes approximately 0.104 second to reach maximum height.

- c. To find the height of the ball as a function of time, use the fact that the derivative of position is velocity, i.e., if $h(t)$ represents the height at time t , then $h'(t) = v(t)$. Because we know $v(t)$ and the initial height, we can form an initial-value problem:

Equation:

$$h'(t) = 2.8369e^{-11.7096t} - 0.8369, \quad h(0) = 1.$$

Integrating both sides of the differential equation with respect to t gives

Equation:

$$\begin{aligned}
 \int h'(t) dt &= \int 2.8369e^{-11.7096t} - 0.8369 dt \\
 h(t) &= -\frac{2.8369}{11.7096} e^{-11.7096t} - 0.8369t + C \\
 h(t) &= -0.2423e^{-11.7096t} - 0.8369t + C.
 \end{aligned}$$

Solve for C by using the initial condition:

Equation:

$$\begin{aligned}h(t) &= -0.2423e^{-11.7096t} - 0.8369t + C \\h(0) &= -0.2423e^{-11.7096(0)} - 0.8369(0) + C \\1 &= -0.2423 + C \\C &= 1.2423.\end{aligned}$$

Therefore

Equation:

$$h(t) = -0.2423e^{-11.7096t} - 0.8369t + 1.2423.$$

After 0.104 second, the height is given by

$$h(0.2) = -0.2423e^{-11.7096t} - 0.8369t + 1.2423 \approx 1.0836 \text{ meter.}$$

Note:

Exercise:

Problem:

The weight of a penny is 2.5 grams (United States Mint, “Coin Specifications,” accessed April 9, 2015, http://www.usmint.gov/about_the_mint/?action=coin_specifications), and the upper observation deck of the Empire State Building is 369 meters above the street. Since the penny is a small and relatively smooth object, air resistance acting on the penny is actually quite small. We assume the air resistance is numerically equal to $0.0025v$. Furthermore, the penny is dropped with no initial velocity imparted to it.

- Set up an initial-value problem that represents the falling penny.
- Solve the problem for $v(t)$.
- What is the terminal velocity of the penny (i.e., calculate the limit of the velocity as t approaches infinity)?

Solution:

- $$\frac{dv}{dt} = -v - 9.8$$

$$v(0) = 0$$
- $$v(t) = 9.8(e^{-t} - 1)$$
- $$\lim_{t \rightarrow \infty} v(t) = \lim_{t \rightarrow \infty} (9.8(e^{-t} - 1)) = -9.8 \text{ m/s} \approx -21.922 \text{ mph}$$

Hint

Set up the differential equation the same way as [\[link\]](#). Remember to convert from grams to kilograms.

Electrical Circuits

A source of electromotive force (e.g., a battery or generator) produces a flow of current in a closed circuit, and this current produces a voltage drop across each resistor, inductor, and capacitor in the circuit. Kirchhoff's Loop Rule states that the sum of the voltage drops across resistors, inductors, and capacitors is equal to the total electromotive force in a closed circuit. We have the following three results:

- The voltage drop across a resistor is given by

Equation:

$$E_R = Ri,$$

where R is a constant of proportionality called the *resistance*, and i is

- the current.
2. The voltage drop across an inductor is given by
Equation:

$$E_L = Li',$$

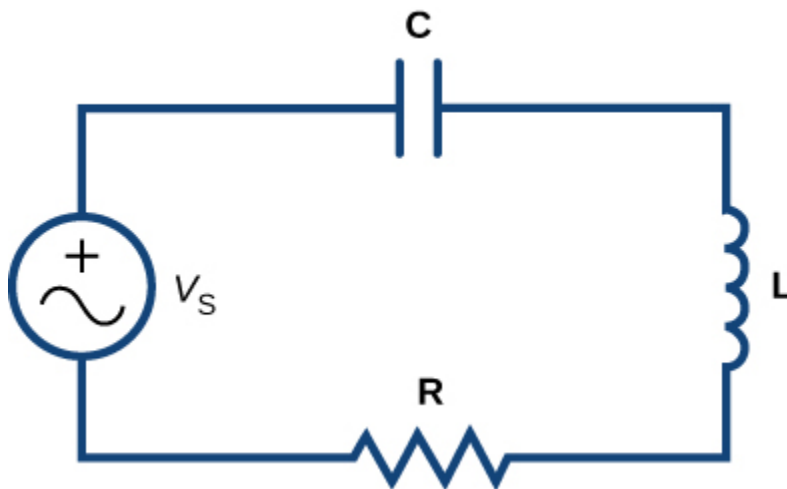
where L is a constant of proportionality called the *inductance*, and i again denotes the current.

3. The voltage drop across a capacitor is given by
Equation:

$$E_C = \frac{1}{C}q,$$

where C is a constant of proportionality called the *capacitance*, and q is the instantaneous charge on the capacitor. The relationship between i and q is $i = q'$.

We use units of volts (V) to measure voltage E , amperes (A) to measure current i , coulombs (C) to measure charge q , ohms (Ω) to measure resistance R , henrys (H) to measure inductance L , and farads (F) to measure capacitance C . Consider the circuit in [\[link\]](#).



A typical electric circuit, containing a voltage generator (V_S), capacitor (C), inductor (L), and resistor (R).

Applying Kirchhoff's Loop Rule to this circuit, we let E denote the electromotive force supplied by the voltage generator. Then

Equation:

$$E_L + E_R + E_C = E.$$

Substituting the expressions for E_L , E_R , and E_C into this equation, we obtain

Equation:

$$Li' + Ri + \frac{1}{C}q = E.$$

If there is no capacitor in the circuit, then the equation becomes

Equation:

$$Li' + Ri = E.$$

This is a first-order differential equation in i . The circuit is referred to as an LR circuit.

Next, suppose there is no inductor in the circuit, but there is a capacitor and a resistor, so $L = 0$, $R \neq 0$, and $C \neq 0$. Then [\[link\]](#) can be rewritten as

Equation:

$$Rq' + \frac{1}{C}q = E,$$

which is a first-order linear differential equation. This is referred to as an RC circuit. In either case, we can set up and solve an initial-value problem.

Example:

Exercise:

Problem:

Finding Current in an RL Electric Circuit

A circuit has in series an electromotive force given by $E = 50 \sin 20t$ V, a resistor of 5Ω , and an inductor of 0.4 H. If the initial current is 0 , find the current at time $t > 0$.

Solution:

We have a resistor and an inductor in the circuit, so we use [\[link\]](#). The voltage drop across the resistor is given by $E_R = Ri = 5i$. The voltage drop across the inductor is given by $E_L = Li' = 0.4i'$. The electromotive force becomes the right-hand side of [\[link\]](#). Therefore [\[link\]](#) becomes

Equation:

$$0.4i' + 5i = 50 \sin 20t.$$

Dividing both sides by 0.4 gives the equation

Equation:

$$i' + 12.5i = 125 \sin 20t.$$

Since the initial current is 0 , this result gives an initial condition of $i(0) = 0$. We can solve this initial-value problem using the five-step strategy for solving first-order differential equations.

Step 1. Rewrite the differential equation as $i' + 12.5i = 125 \sin 20t$. This gives $p(t) = 12.5$ and $q(t) = 125 \sin 20t$.

Step 2. The integrating factor is $\mu(t) = e^{\int 12.5 dt} = e^{12.5t}$.

Step 3. Multiply the differential equation by $\mu(t)$:

Equation:

$$\begin{aligned} e^{12.5t} i' + 12.5 e^{12.5t} i &= 125 e^{12.5t} \sin 20t \\ \frac{d}{dt} [i e^{12.5t}] &= 125 e^{12.5t} \sin 20t. \end{aligned}$$

Step 4. Integrate both sides:

Equation:

$$\begin{aligned} \int \frac{d}{dt} [i e^{12.5t}] dt &= \int 125 e^{12.5t} \sin 20t dt \\ i e^{12.5t} &= \left(\frac{250 \sin 20t - 400 \cos 20t}{89} \right) e^{12.5t} + C \\ i(t) &= \frac{250 \sin 20t - 400 \cos 20t}{89} + C e^{-12.5t}. \end{aligned}$$

Step 5. Solve for C using the initial condition $v(0) = 2$:

Equation:

$$\begin{aligned} i(t) &= \frac{250 \sin 20t - 400 \cos 20t}{89} + C e^{-12.5t} \\ i(0) &= \frac{250 \sin 20(0) - 400 \cos 20(0)}{89} + C e^{-12.5(0)} \\ 0 &= -\frac{400}{89} + C \\ C &= \frac{400}{89}. \end{aligned}$$

Therefore the solution to the initial-value problem is

$$i(t) = \frac{250 \sin 20t - 400 \cos 20t + 400 e^{-12.5t}}{89} = \frac{250 \sin 20t - 400 \cos 20t}{89} + \frac{400 e^{-12.5t}}{89}.$$

The first term can be rewritten as a single cosine function. First, multiply and divide by $\sqrt{250^2 + 400^2} = 50\sqrt{89}$:

Equation:

$$\begin{aligned}\frac{250\sin 20t - 400\cos 20t}{89} &= \frac{50\sqrt{89}}{89} \left(\frac{250\sin 20t - 400\cos 20t}{50\sqrt{89}} \right) \\ &= -\frac{50\sqrt{89}}{89} \left(\frac{8\cos 20t}{\sqrt{89}} - \frac{5\sin 20t}{\sqrt{89}} \right).\end{aligned}$$

Next, define φ to be an acute angle such that $\cos \varphi = \frac{8}{\sqrt{89}}$. Then $\sin \varphi = \frac{5}{\sqrt{89}}$ and

Equation:

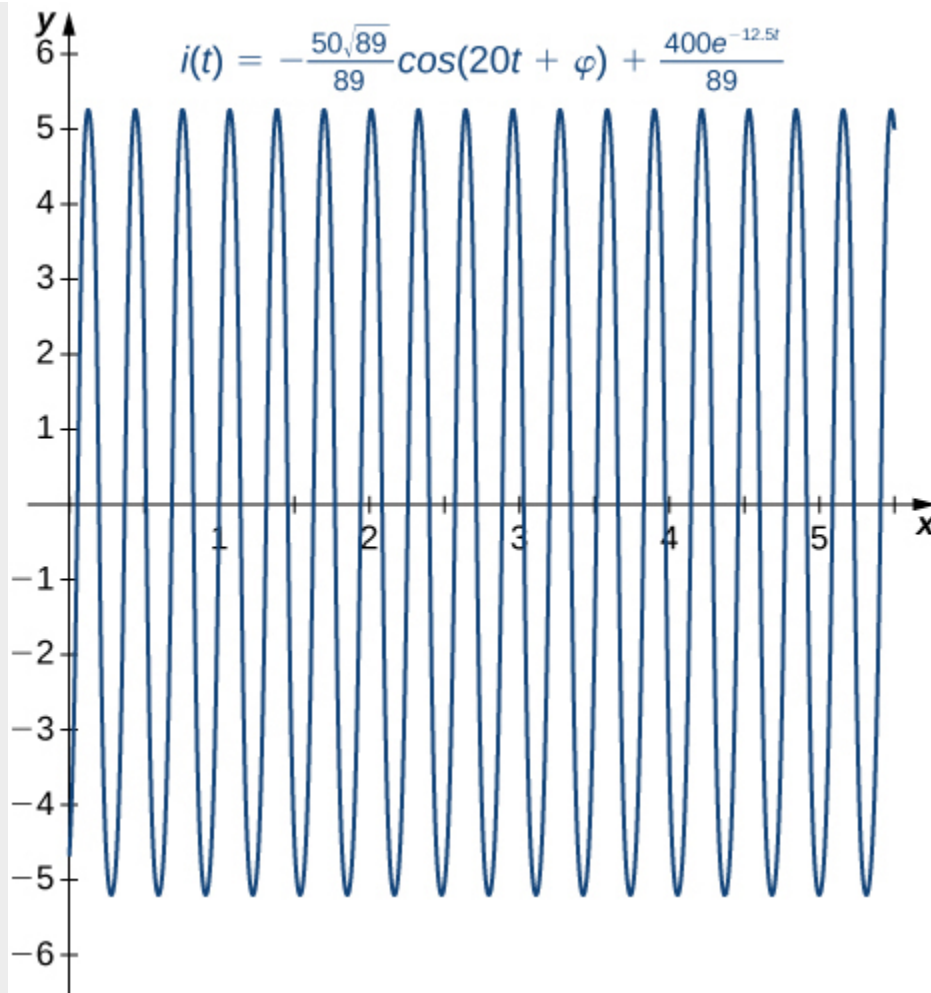
$$\begin{aligned}-\frac{50\sqrt{89}}{89} \left(\frac{8\cos 20t}{\sqrt{89}} - \frac{5\sin 20t}{\sqrt{89}} \right) &= -\frac{50\sqrt{89}}{89} (\cos \varphi \cos 20t - \sin \varphi \sin 20t) \\ &= -\frac{50\sqrt{89}}{89} \cos (20t + \varphi).\end{aligned}$$

Therefore the solution can be written as

Equation:

$$i(t) = -\frac{50\sqrt{89}}{89} \cos (20t + \varphi) + \frac{400e^{-12.5t}}{89}.$$

The second term is called the *attenuation* term, because it disappears rapidly as t grows larger. The phase shift is given by φ , and the amplitude of the steady-state current is given by $\frac{50\sqrt{89}}{89}$. The graph of this solution appears in [\[link\]](#):



Note:

Exercise:

Problem:

A circuit has in series an electromotive force given by $E = 20 \sin 5t$ V, a capacitor with capacitance 0.02 F, and a resistor of 8Ω . If the initial charge is 4 C, find the charge at time $t > 0$.

Solution:

Initial-value problem:

$$8q' + \frac{1}{0.02}q = 20\sin 5t, \quad q(0) = 4$$

$$q(t) = \frac{10\sin 5t - 8\cos 5t + 172e^{-6.25t}}{41}$$

Hint

Use [\[link\]](#) for an RC circuit to set up an initial-value problem.

Key Concepts

- Any first-order linear differential equation can be written in the form $y' + p(x)y = q(x)$.
- We can use a five-step problem-solving strategy for solving a first-order linear differential equation that may or may not include an initial value.
- Applications of first-order linear differential equations include determining motion of a rising or falling object with air resistance and finding current in an electrical circuit.

Key Equations

- **standard form**
 $y' + p(x)y = q(x)$
- **integrating factor**
 $\mu(x) = e^{\int p(x) dx}$

Are the following differential equations linear? Explain your reasoning.

Exercise:

Problem: $\frac{dy}{dx} = x^2y + \sin x$

Exercise:

Problem: $\frac{dy}{dt} = ty$

Solution:

Yes

Exercise:

Problem: $\frac{dy}{dt} + y^2 = x$

Exercise:

Problem: $y' = x^3 + e^x$

Solution:

Yes

Exercise:

Problem: $y' = y + e^y$

Write the following first-order differential equations in standard form.

Exercise:

Problem: $y' = x^3y + \sin x$

Solution:

$$y' - x^3y = \sin x$$

Exercise:

Problem: $y' + 3y - \ln x = 0$

Exercise:

Problem: $-xy' = (3x + 2)y + xe^x$

Solution:

$$y' + \frac{(3x+2)}{x}y = -e^x$$

Exercise:

Problem: $\frac{dy}{dt} = 4y + ty + \tan t$

Exercise:

Problem: $\frac{dy}{dt} = yx(x+1)$

Solution:

$$\frac{dy}{dt} - yx(x+1) = 0$$

What are the integrating factors for the following differential equations?

Exercise:

Problem: $y' = xy + 3$

Exercise:

Problem: $y' + e^x y = \sin x$

Solution:

$$e^x$$

Exercise:

Problem: $y' = x \ln(x)y + 3x$

Exercise:

Problem: $\frac{dy}{dx} = \tanh(x)y + 1$

Solution:

$$-\ln(\cosh x)$$

Exercise:

Problem: $\frac{dy}{dt} + 3ty = e^t y$

Solve the following differential equations by using integrating factors.

Exercise:

Problem: $y' = 3y + 2$

Solution:

$$y = Ce^{3x} - \frac{2}{3}$$

Exercise:

Problem: $y' = 2y - x^2$

Exercise:

Problem: $xy' = 3y - 6x^2$

Solution:

$$y = Cx^3 + 6x^2$$

Exercise:

Problem: $(x + 2)y' = 3x + y$

Exercise:

Problem: $y' = 3x + xy$

Solution:

$$y = Ce^{x^2/2} - 3$$

Exercise:

Problem: $xy' = x + y$

Exercise:

Problem: $\sin(x)y' = y + 2x$

Solution:

$$y = C \tan\left(\frac{x}{2}\right) - 2x + 4 \tan\left(\frac{x}{2}\right) \ln\left(\sin\left(\frac{x}{2}\right)\right)$$

Exercise:

Problem: $y' = y + e^x$

Exercise:

Problem: $xy' = 3y + x^2$

Solution:

$$y = Cx^3 - x^2$$

Exercise:

Problem: $y' + \ln x = \frac{y}{x}$

Solve the following differential equations. Use your calculator to draw a family of solutions. Are there certain initial conditions that change the behavior of the solution?

Exercise:

Problem: [T] $(x + 2)y' = 2y - 1$

Solution:

$$y = C(x + 2)^2 + \frac{1}{2}$$

Exercise:

Problem: [T] $y' = 3e^{t/3} - 2y$

Exercise:

Problem: [T] $xy' + \frac{y}{2} = \sin(3t)$

Solution:

$$y = \frac{C}{\sqrt{x}} + 2\sin(3t)$$

Exercise:

Problem: [T] $xy' = 2\frac{\cos x}{x} - 3y$

Exercise:

Problem: [T] $(x + 1)y' = 3y + x^2 + 2x + 1$

Solution:

$$y = C(x + 1)^3 - x^2 - 2x - 1$$

Exercise:

Problem: [T] $\sin(x)y' + \cos(x)y = 2x$

Exercise:

Problem: [T] $\sqrt{x^2 + 1}y' = y + 2$

Solution:

$$y = Ce^{\sinh^{-1}x} - 2$$

Exercise:

Problem: [T] $x^3y' + 2x^2y = x + 1$

Solve the following initial-value problems by using integrating factors.

Exercise:

Problem: $y' + y = x, y(0) = 3$

Solution:

$$y = x + 4e^x - 1$$

Exercise:

Problem: $y' = y + 2x^2, y(0) = 0$

Exercise:

Problem: $xy' = y - 3x^3, y(1) = 0$

Solution:

$$y = -\frac{3x}{2}(x^2 - 1)$$

Exercise:

Problem: $x^2y' = xy - \ln x, y(1) = 1$

Exercise:

Problem: $(1 + x^2)y' = y - 1, y(0) = 0$

Solution:

$$y = 1 - e^{\tan^{-1}x}$$

Exercise:

Problem: $xy' = y + 2x \ln x, y(1) = 5$

Exercise:

Problem: $(2 + x)y' = y + 2 + x, y(0) = 0$

Solution:

$$y = (x + 2) \ln \left(\frac{x+2}{2} \right)$$

Exercise:

Problem: $y' = xy + 2xe^x, y(0) = 2$

Exercise:

Problem: $\sqrt{x}y' = y + 2x, y(0) = 1$

Solution:

$$y = 2e^{2\sqrt{x}} - 2x - 2\sqrt{x} - 1$$

Exercise:

Problem: $y' = 2y + xe^x, y(0) = -1$

Exercise:

Problem:

A falling object of mass m can reach terminal velocity when the drag force is proportional to its velocity, with proportionality constant k . Set up the differential equation and solve for the velocity given an initial velocity of 0.

Solution:

$$v(t) = \frac{gm}{k} (1 - e^{-kt/m})$$

Exercise:**Problem:**

Using your expression from the preceding problem, what is the terminal velocity? (*Hint*: Examine the limiting behavior; does the velocity approach a value?)

Exercise:**Problem:**

[T] Using your equation for terminal velocity, solve for the distance fallen. How long does it take to fall 5000 meters if the mass is 100 kilograms, the acceleration due to gravity is 9.8 m/s^2 and the proportionality constant is 4?

Solution:

40.451 seconds

Exercise:**Problem:**

A more accurate way to describe terminal velocity is that the drag force is proportional to the square of velocity, with a proportionality constant k . Set up the differential equation and solve for the velocity.

Exercise:

Problem:

Using your expression from the preceding problem, what is the terminal velocity? (*Hint*: Examine the limiting behavior: Does the velocity approach a value?)

Solution:

$$\sqrt{\frac{gm}{k}}$$

Exercise:**Problem:**

[T] Using your equation for terminal velocity, solve for the distance fallen. How long does it take to fall 5000 meters if the mass is 100 kilograms, the acceleration due to gravity is 9.8 m/s^2 and the proportionality constant is 4? Does it take more or less time than your initial estimate?

For the following problems, determine how parameter a affects the solution.

Exercise:**Problem:**

Solve the generic equation $y' = ax + y$. How does varying a change the behavior?

Solution:

$$y = Ce^x - a(x + 1)$$

Exercise:**Problem:**

Solve the generic equation $y' = ax + y$. How does varying a change the behavior?

Exercise:**Problem:**

Solve the generic equation $y' = ax + xy$. How does varying a change the behavior?

Solution:

$$y = Ce^{x^2/2} - a$$

Exercise:**Problem:**

Solve the generic equation $y' = x + axy$. How does varying a change the behavior?

Exercise:**Problem:**

Solve $y' - y = e^{kt}$ with the initial condition $y(0) = 0$. As k approaches 1, what happens to your formula?

Solution:

$$y = \frac{e^{kt} - e^t}{k - 1}$$

Chapter Review Exercises

True or False? Justify your answer with a proof or a counterexample.

Exercise:

Problem: The differential equation $y' = 3x^2y - \cos(x)y''$ is linear.

Exercise:

Problem: The differential equation $y' = x - y$ is separable.

Solution:

F

Exercise:**Problem:**

You can explicitly solve all first-order differential equations by separation or by the method of integrating factors.

Exercise:**Problem:**

You can determine the behavior of all first-order differential equations using directional fields or Euler's method.

Solution:

T

For the following problems, find the general solution to the differential equations.

Exercise:

Problem: $y' = x^2 + 3e^x - 2x$

Exercise:

Problem: $y' = 2^x + \cos^{-1}x$

Solution:

$$y(x) = \frac{2^x}{\ln(2)} + x \cos^{-1}x - \sqrt{1-x^2} + C$$

Exercise:

Problem: $y' = y(x^2 + 1)$

Exercise:

Problem: $y' = e^{-y} \sin x$

Solution:

$$y(x) = \ln(C - \cos x)$$

Exercise:

Problem: $y' = 3x - 2y$

Exercise:

Problem: $y' = y \ln y$

Solution:

$$y(x) = e^{e^{C+x}}$$

For the following problems, find the solution to the initial value problem.

Exercise:

Problem: $y' = 8x - \ln x - 3x^4, y(1) = 5$

Exercise:

Problem: $y' = 3x - \cos x + 2, y(0) = 4$

Solution:

$$y(x) = 4 + \frac{3}{2}x^2 + 2x - \sin x$$

Exercise:

Problem: $xy' = y(x - 2), y(1) = 3$

Exercise:

Problem: $y' = 3y^2(x + \cos x), y(0) = -2$

Solution:

$$y(x) = -\frac{2}{1+3(x^2+2\sin x)}$$

Exercise:

Problem: $(x-1)y' = y-2, y(0) = 0$

Exercise:

Problem: $y' = 3y - x + 6x^2, y(0) = -1$

Solution:

$$y(x) = -2x^2 - 2x - \frac{1}{3} - \frac{2}{3}e^{3x}$$

For the following problems, draw the directional field associated with the differential equation, then solve the differential equation. Draw a sample solution on the directional field.

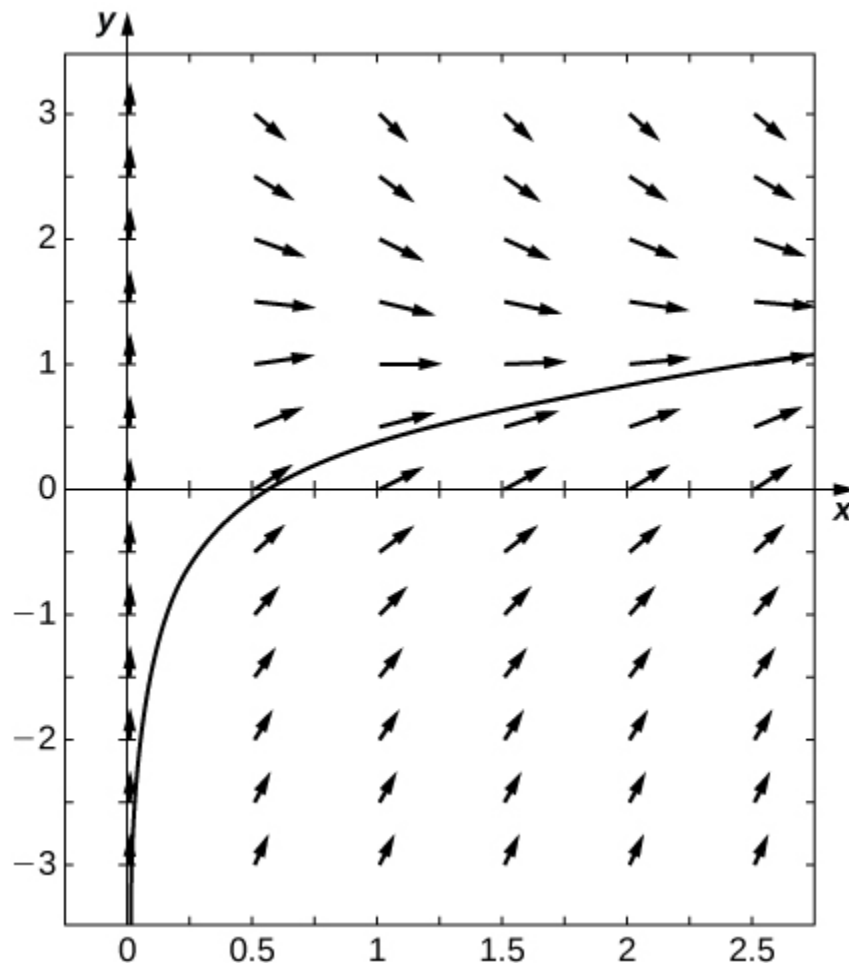
Exercise:

Problem: $y' = 2y - y^2$

Exercise:

Problem: $y' = \frac{1}{x} + \ln x - y, \text{ for } x > 0$

Solution:



$$y(x) = Ce^{-x} + \ln x$$

For the following problems, use Euler's Method with $n = 5$ steps over the interval $t = [0, 1]$. Then solve the initial-value problem exactly. How close is your Euler's Method estimate?

Exercise:

Problem: $y' = -4yx, y(0) = 1$

Exercise:

Problem: $y' = 3^x - 2y, y(0) = 0$

Solution:

Euler: 0.6939, exact solution: $y(x) = \frac{3^x - e^{-2x}}{2 + \ln(3)}$

For the following problems, set up and solve the differential equations.

Exercise:

Problem:

A car drives along a freeway, accelerating according to $a = 5 \sin(\pi t)$, where t represents time in minutes. Find the velocity at any time t , assuming the car starts with an initial speed of 60 mph.

Exercise:

Problem:

You throw a ball of mass 2 kilograms into the air with an upward velocity of 8 m/s. Find exactly the time the ball will remain in the air, assuming that gravity is given by $g = 9.8 \text{ m/s}^2$.

Solution:

$\frac{40}{49}$ second

Exercise:

Problem:

You drop a ball with a mass of 5 kilograms out an airplane window at a height of 5000 m. How long does it take for the ball to reach the ground?

Exercise:

Problem:

You drop the same ball of mass 5 kilograms out of the same airplane window at the same height, except this time you assume a drag force proportional to the ball's velocity, using a proportionality constant of 3 and the ball reaches terminal velocity. Solve for the distance fallen as a function of time. How long does it take the ball to reach the ground?

Solution:

$$x(t) = 5000 + \frac{245}{9} - \frac{49}{3}t - \frac{245}{9}e^{-5/3t}, t = 307.8 \text{ seconds}$$

Exercise:**Problem:**

A drug is administered to a patient every 24 hours and is cleared at a rate proportional to the amount of drug left in the body, with proportionality constant 0.2. If the patient needs a baseline level of 5 mg to be in the bloodstream at all times, how large should the dose be?

Exercise:**Problem:**

A 1000-liter tank contains pure water and a solution of 0.2 kg salt/L is pumped into the tank at a rate of 1 L/min and is drained at the same rate. Solve for total amount of salt in the tank at time t .

Solution:

$$T(t) = 200 (1 - e^{-t/1000})$$

Exercise:**Problem:**

You boil water to make tea. When you pour the water into your teapot, the temperature is 100°C . After 5 minutes in your 15°C room, the temperature of the tea is 85°C . Solve the equation to determine the temperatures of the tea at time t . How long must you wait until the tea is at a drinkable temperature (72°C)?

Exercise:

Problem:

The human population (in thousands) of Nevada in 1950 was roughly 160. If the carrying capacity is estimated at 10 million individuals, and assuming a growth rate of 2% per year, develop a logistic growth model and solve for the population in Nevada at any time (use 1950 as time = 0). What population does your model predict for 2000? How close is your prediction to the true value of 1,998,257?

Solution:

$$P(t) = \frac{1600000e^{0.02t}}{9840 + 160e^{0.02t}}$$

Exercise:**Problem:**

Repeat the previous problem but use Gompertz growth model. Which is more accurate?

Glossary

integrating factor

any function $f(x)$ that is multiplied on both sides of a differential equation to make the side involving the unknown function equal to the derivative of a product of two functions

linear

description of a first-order differential equation that can be written in the form $a(x)y' + b(x)y = c(x)$

standard form

the form of a first-order linear differential equation obtained by writing the differential equation in the form $y' + p(x)y = q(x)$